Exercise 1.1 (4 points) *Lie algebra of SU(3)*

In the 3-dimensional defining representation of SU(3) the generators $T^a = \lambda^a/2$ are given by the Gell-Mann matrices λ^a $(a = 1, \ldots, 8)$:

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

$$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The matrices λ^a obey the relations

$$[\lambda^a, \lambda^b] = 2if^{abc}\lambda^c, \qquad \lambda^a = (\lambda^a)^\dagger, \qquad \mathrm{Sp}(\lambda^a) = 0, \qquad \mathrm{Sp}(\lambda^a\lambda^b) = 2\delta^{ab}$$

with f^{abc} being the structure constants of SU(3).

a) Calculate the matrices $A = \sum_{a=1}^{3}(\lambda^a)^2$ and $B = \sum_{a=1}^{8}(\lambda^a)^2$. Are A and B Casimir operators?

b) Explain why the anticommutator of λ^a can be written as $\{\lambda^a, \lambda^b\} = C\delta^{ab}\mathbf{1} + 2d^{abc}\lambda^c$ with the real constants C and d^{abc}. Determine C.

c) Prove

$$\mathrm{Sp}(\lambda^a[\lambda^b, \lambda^c]) = 4if^{bca}, \qquad \mathrm{Sp}(\lambda^a\{\lambda^b, \lambda^c\}) = 4d^{bca}.$$

From these relations, deduce the complete antisymmetry of f^{abc} and the complete symmetry of d^{abc} with respect to the intechange of two of the indices a, b, c.

Please turn over!

Exercise 1.2 (3 points) *SU(2) gauge theory*

Let the fields A^a_μ with $a = 1, 2, 3$ denote a triplet of gauge bosons of a Yang-Mills theory with gauge group SU(2). Formulate the transformation behaviour of the fields

$$A_\mu(x) = A^3_\mu(x), \qquad W^\pm_\mu(x) = \frac{1}{\sqrt{2}}[A^1_\mu(x) \mp iA^2_\mu(x)]$$

under the specific gauge transformation $U_3 = \exp\{-igT^3\omega^3(x)\}$. What is the meaning of the fields A_μ and $W^\pm_\mu$ if g and T^3 are identified with the elementary charge e and the operator $\hat{Q}$ of the electromagnetic charge, respectively?

Exercise 1.3 (3 points) *Electrodynamics of a scalar field*

Given a complex scalar field ϕ characterizing a spin-0 boson with electric charge Qe and mass m, the free motion and the self-interaction of ϕ are described by

$$\mathcal{L}_\phi(\phi, \partial_\mu\phi) = (\partial_\mu\phi)^*(\partial^\mu\phi) - m^2\phi^*\phi - V(\phi^*\phi),$$

where the potential V is not further specified.

a) Implement the electromagnetic interaction into this theory by applying minimal substitution analogous to the procedure in QED with fermions.

b) $\mathcal{L}_\phi$ is invariant under the global phase transformation $\phi \to \phi' = \exp\{-iQe\omega\}\phi$. What is the conserved current j_μ corresponding to this symmetry? Does the current j_μ change its form if the photon field A_μ is switched on and off, respectively?