Exercises to Advanced Quantum Mechanics — Sheet 12
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Exercise 12.1 *Charged-particle scattering off a charge distribution* (4 points)

We consider the scattering of a point-like, spinless particle with electric charge $Q_1 e$ and mass M off a smoothly distributed charge, whose spherically symmetric charge density $\rho(r)$ is centered around the origin at $r = 0$.

a) Show that the electric potential $\Phi(r)$ of the charge distribution can be written as

$$\Phi(r) = \frac{1}{r}\int_0^r \mathrm{d}r'\, r'^2 \frac{\rho(r')}{\epsilon_0} + \int_r^\infty \mathrm{d}r'\, r' \,\frac{\rho(r')}{\epsilon_0}. \tag{1}$$

b) Assuming first that the total charge $Q_2 e = 4\pi \int_0^\infty \mathrm{d}r'\, r'^2 \rho(r')$ of the distribution vanishes, calculate the scattering amplitude $f_k(\Omega)$ in Born approximation, where k is the wavenumber of the incoming particles. Express the result in terms of the *form factor* $F(q)$ of the charge distribution, which is defined by

$$F(q) = \int \mathrm{d}^3x\, \mathrm{e}^{-\mathrm{i}\vec{q}\cdot\vec{x}}\, \frac{\rho(r)}{e}, \qquad r = |\vec{x}|, \quad q = |\vec{q}|. \tag{2}$$

You should find that $f_k(\Omega) = \text{const.} \times \frac{F(q)}{q^2}$ with $q = 2k\sin\left(\frac{\theta}{2}\right)$.

c) Derive the corresponding differential cross section $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ and identify its relation to the Rutherford cross section. Argue why the result is valid for $Q_2 \neq 0$ as well. What is the limit of $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ for $ka \ll 1$ if $Q_2 \neq 0$ and a is the radius of a sphere containing the entire charge distribution?

d) Derive the differential Born cross section for a homogeneous charge distribution contained in a sphere of radius a.

Rutherford scattered α particles with a velocity of $v = 1.5 \cdot 10^7\,\mathrm{m/s}$ off gold nuclei and did not find a significant deviation from the cross section for a point-like nucleus charge. Interpreting a as the radius of the nucleus in the model of a homogeneous charge distribution in the nucleus, derive an upper bound on the radius based on the condition that the second maximum of the differential cross section is visible, i.e. appears at a scattering angle $\theta \leq 90°$.

Please turn over!

Exercise 12.2 *Scattering off hard spheres* (3 points)

We consider the scattering of particles of mass M at the potential

$$V(r) = \begin{cases} 0, & r \geq a \\ \infty, & r < a \end{cases}, \tag{3}$$

of a "hard sphere".

a) Determine $\tan \delta_l(k)$ of the scattering phases $\delta_l(k)$ from the exact solution of the Schrödinger equation.

b) Calculate the leading behaviour of the differential and total cross sections in the low-energy limit $ka \ll 1$, keeping terms up to order k^2.

c) Show that the total cross section at high energies ($ka \gg 1$) is approximately given by $2\pi a^2$. Compare this result with the one obtained in classical mechanics.

   Hint: As argued in the lecture, only partial waves with $l \lesssim ka$ are relevant. Moreover, in good approximation, the asymptotics of $j_l(\rho)$ and $n_l(\rho)$ for large ρ can be used.