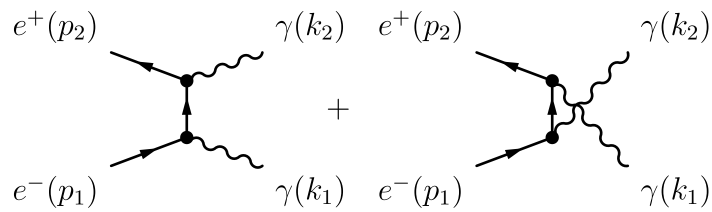
Exercises "Modern methods of Quantum Chromodynamics" **WS 14/15**

Problem 5 (3 Points) *$e^-e^+ \to \gamma\gamma$ and gauge invariance*

Consider the process $e^-(p_1)e^+(p_2) \to \gamma(k_1)\gamma(k_2)$, which receives contributions from two diagrams at leading order:



a) Write down the analytic expression of the scattering amplitude.

b) Verify that the amplitude is invariant under the replacement

$$\epsilon^\mu(k) \to \epsilon^\mu(k) + a\,k^\mu$$

for one of the photon-polarization vectors, provided the electron and positron spinors satisfy the Dirac equation.

Problem 6 (2 Points) *Lie-algebra representations*

The generators $\mathbf{T}^{(R),a}$ of a representation R of a Lie algebra with structure constants f^{abc} satisfy the commutation relations

$$[\mathbf{T}^{(R)a}, \mathbf{T}^{(R)b}] = \mathrm{i} f^{abc}\mathbf{T}^{(R)c}.$$

a) Show that the matrices $(T^{(\mathrm{ad})a})_{bc} = -\mathrm{i}f^{abc}$ form a representation as a result of the Jacobi identity.

b) Given a representation R, show that the generators $\mathbf{T}^{(\bar{R})a} = -\mathbf{T}^{(R)a,T}$ also form a representation.

Problem 7 (3 Points) *Yang Mills theory*

The Lagrangian of QCD for a single quark flavour is given by

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} + \bar{Q} \left(\mathrm{i} \not{D} - m_q \right) Q$$

with the field strength tensor $F^a_{\mu\nu} = (\partial_\mu A^a - \partial_\nu A^a_\mu) - g_s f^{abc} A^b_\mu A^c_\nu$ and the covariant derivative $D_\mu = \partial_\mu + \mathrm{i} g_s T^a A^a_\mu$.

a) Show that the equations of motion of QCD are given by

$$\begin{aligned} (\mathrm{i}\not{\partial} - m_q) Q &= g_s T^a \not{A}^a Q, \\ \mathrm{i}(\partial_\mu \bar{Q})\gamma^\mu + m_q \bar{Q} &= -g_s \bar{Q} \not{T}^a A^a, \\ D^{\text{ad}}_{ab,\mu} F^{a,\mu\nu} &= j^{a,\nu} \qquad \text{with} \quad j^{a,\mu} = g_s\, \bar{Q} \gamma^\mu T^a Q. \end{aligned}$$

where the covariant derivative in the adjoint representation is given by $D^{\text{ad}}_{ab,\mu} = \partial_\mu \delta_{ab} + g_s f^{abc} A^c_\mu$.

b) Show that the quark current $j^{\mu,a}$ is covariantly conserved, $D^{\text{ad}}_{ab,\mu} j^{b,\mu} = 0$, for solutions of the equations of motion.

Problem 8 (2 Points) *Wilson lines and covariant derivatives*

A quark field $Q(x)$ transforms under a $SU(3)$ gauge transformation as $Q(x) \to Q'(x) = U(x)Q(x)$. A *Wilson line* is a function $W(x, y)$ that transforms under gauge transformations as

$$W(x, y) \to W'(x, y) = U(x)W(x, y)U^\dagger(y)$$

so that $W(x, y)Q(y) \to U(x)W(x, y)Q(y)$. The introduction of Wilson lines allows to interpret the transformations of the gauge fields and the definition of the covariant derivative in a geometrical way. For $y = x + \delta x$ the gauge field can be defined as the coefficient of the term linear in δx:

$$W(x, x + \delta x) = 1 + \mathrm{i} g_s\, A^a_\mu(x) T^a\, \delta x^\mu + \mathcal{O}(\delta x^2).$$

a) Derive the behaviour of the gauge field under infinitesimal gauge transformations $U(x) = 1 - \mathrm{i} g_s \delta\omega^a(x) T^a + \ldots$.

b) The *covariant derivative* of a quark field along the direction n^μ is defined as

$$n^\mu D_\mu Q(x) = \lim_{\epsilon \to 0} \frac{W(x, x + \epsilon n) Q(x + \epsilon n) - Q(x)}{\epsilon}.$$

Verify that this definition reproduces the expression for the covariant derivative given in the lecture.