Problem 9 (4 Points) *Colour algebra*

a) Proof the identity for the generators T^a of $SU(N)$:

$$T^{a,i}{}_j T^{a,k}{}_l = \frac{1}{2}\left(\delta^i{}_l \delta^k{}_j - \frac{1}{N}\delta^i{}_j \delta^k{}_l\right).$$

Use that the generators T^a and the unit matrix form a basis of the hermitian $N \times N$ matrices. The normalization of the generators is $\mathrm{Tr}(T^a T^b) = T_F \delta_{ab}$ with $T_F = \frac{1}{2}$.

b) Compute the trace

$$\mathrm{Tr}(T^a T^b T^a T^b).$$

c) Argue that the anti-commutator of the generators can be written as

$$\{T^a, T^b\} = C\delta^{ab}\mathbf{1} + d^{abc}T^c$$

with real constants C and d^{abc}. Compute C.

d) The generators of the adjoint representation are given in terms of the structure constants by $(T^{(\mathrm{ad})a})_{bc} = -\mathrm{i}f^{abc}$. Compute the Casimir-operator

$$\left(T^{(\mathrm{ad})a}T^{(\mathrm{ad})a}\right)_{bc} = C_A \delta_{bc}$$

in the adjoint representation.
(Hint: Compute $\mathrm{Tr}\left\{[T^a, T^b][T^a, T^c]\right\}$ in two different ways)

Problem 10 (3 Points) *Colour-octet scalars*

Consider a set of eight real scalar fields $\phi_a(x)$ with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_a)(\partial_\mu \phi_a) - \frac{m^2}{2}\phi_a^2 - \lambda(\phi_a\phi_a)^2$$

a) Introduce an interaction of the scalars with the gluon field $A^a_\mu(x)$ in such a way that the Lagrangian is invariant under the local transformations

$$\phi(x) \to U^{(\mathrm{ad})}(x)\phi(x),$$
$$T^{a(ad)}A^a_\mu(x) \to U^{(\mathrm{ad})}(x)T^{a(ad)}A^a_\mu(x)U^{(\mathrm{ad})\dagger}(x) + \frac{\mathrm{i}}{g_s}(\partial_\mu U^{(\mathrm{ad})}(x))T^{a(ad)}U^{(\mathrm{ad})\dagger}(x)$$

with the transformation in the adjoint representation,

$$U^{(\mathrm{ad})}(x) = \exp(-\mathrm{i}g_s\omega^a(x)T^{(\mathrm{ad})a}).$$

b) Give the Feynman rules for the interactions of the scalars.

c) *Bonus question:* Are there other gauge invariant quartic scalar interactions in addition to the term $(\phi_a^\dagger \phi_a)^2$? (1 bonus point)

Problem 11 (3 Points) *Deep inelastic scattering*

The hadronic tensor $W^{\mu\nu}(p,q)$ in (unpolarized) deep inelastic scattering $e^-(k)P(p) \to e^-(k') + X$ with $q = k - k'$ satisfies the properties

$$q_\mu W^{\mu\nu}(p,q) = 0\,, \qquad\qquad W^{\mu\nu}(q,p), = W^{\nu\mu}(q,p).$$

a) Show that these properties imply that the hadronic tensor can be expressed in terms of two scalar coefficient functions $F_{1/2}$ in the form

$$W_{\mu\nu}(q,p) = F_1\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + \frac{F_2}{(p\cdot q)}\left(p^\mu - q^\mu\frac{(p\cdot q)}{q^2}\right)\left(p^\nu - q^\nu\frac{(p\cdot q)}{q^2}\right).$$

b) Compute the contraction $W_{\mu\nu}L^{\mu\nu}$ with the leptonic tensor

$$L^{\mu\nu} = k^\mu k'^\nu + k'^\mu k^\nu - (k\cdot k')g^{\mu\nu}.$$