Exercise 1.1 *Some properties of Lorentz transformations* (2 points)

Lorentz transformations, which transform a four-vector $a^\mu = (a^0, \vec{a})$ to $a'^\mu = \Lambda^\mu{}_\nu a^\nu$, comprise all 4×4 matrices Λ that leave the metric tensor $g^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$ invariant, i.e. $g^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g^{\alpha\beta}$. In the following we consider the "proper orthochronous Lorentz group" $L_+^\uparrow$ that comprises all such Λ with the two constraints that $\det\Lambda = +1$ and $\Lambda^0{}_0 > 0$. The group $L_+^\uparrow$ consists of all rotations in space and "boosts", which relate two frames of reference with a non-vanishing relative velocity.

a) A boost with relative velocity $\vec{v} = (v^1, v^2, v^3)$ is described by the Λ matrix of the form

$$L(v^1, v^2, v^3)^\mu{}_\nu = \begin{pmatrix} \gamma & -v^1\gamma & -v^2\gamma & -v^3\gamma \\ -v^1\gamma & 1 + \frac{v^1 v^1}{\vec{v}^2}(\gamma - 1) & \frac{v^1 v^2}{\vec{v}^2}(\gamma - 1) & \frac{v^1 v^3}{\vec{v}^2}(\gamma - 1) \\ -v^2\gamma & \frac{v^2 v^1}{\vec{v}^2}(\gamma - 1) & 1 + \frac{v^2 v^2}{\vec{v}^2}(\gamma - 1) & \frac{v^2 v^3}{\vec{v}^2}(\gamma - 1) \\ -v^3\gamma & \frac{v^3 v^1}{\vec{v}^2}(\gamma - 1) & \frac{v^3 v^2}{\vec{v}^2}(\gamma - 1) & 1 + \frac{v^3 v^3}{\vec{v}^2}(\gamma - 1) \end{pmatrix},$$

where $\gamma = 1/\sqrt{1 - \vec{v}^2}$. Calculate the boosted components x'^μ for the four-vectors $x_{||}^\mu = (x^0, r\vec{e})$ and $x_\perp^\mu = (x^0, r\vec{e}_\perp)$ whose directions in space are parallel and perpendicular to the direction $\vec{e} = \vec{v}/|\vec{v}|$ of the relative velocity, respectively, i.e. $\vec{e}_\perp \cdot \vec{e} = 0$.

b) Show that the sign of the time-like component a^0 of any non-space-like four-vector a^μ (i.e. $a^2 \geq 0$) is invariant under all Lorentz transformations $\Lambda \in L_+^\uparrow$.

c) Calculate $W = L_2(0, -v^2, 0) L_1(-v^1, 0, 0) L_2(0, v^2, 0) L_1(v^1, 0, 0)$ for small velocities v^k and keep terms up to quadratic order in products of components v^k. What kind of transformation is described by W?

d) Show that the totally antisymmetric tensor

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } (\mu\nu\rho\sigma) = \text{even permutation of } (0123), \\ -1 & \text{if } (\mu\nu\rho\sigma) = \text{odd permutation of } (0123), \\ 0 & \text{otherwise.} \end{cases}$$

is an invariant tensor under all $\Lambda \in L_+^\uparrow$, i.e. $\epsilon'^{\mu\nu\rho\sigma} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda^\rho{}_\gamma \Lambda^\sigma{}_\delta \epsilon^{\alpha\beta\gamma\delta}$.

e) Show that $d^\mu = \epsilon^{\mu\nu\rho\sigma} a_\nu b_\rho c_\sigma$ transforms like a four-vector under $\Lambda \in L_+^\uparrow$ if a^μ, b^μ, and c^μ are four-vectors.

Please turn over!

Exercise 1.2 *Kinematics of a $1 \to 2$ particle decay* (2 points)

A particle of mass M and four-momentum k^μ decays into two particles of masses m_i and four-momenta p_i^μ $(i = 1, 2)$. The momenta obey their mass-shell conditions $k^2 = M^2$ and $p_i^2 = m_i^2$ and, in the centre-of-mass frame Σ, are given by

$$k^\mu = (M, \mathbf{0}), \qquad p_i^\mu = (E_i, |\mathbf{p}_i| \cos\phi_i \sin\theta_i, |\mathbf{p}_i| \sin\phi_i \sin\theta_i, |\mathbf{p}_i| \cos\theta_i).$$

a) What are the consequences of four-momentum conservation $k = p_1 + p_2$ for the energies E_i, for the absolute values $|\mathbf{p}_i|$ of the three-momenta and for the angles θ_i, ϕ_i?

b) Calculate E_i and $|\mathbf{p}_i|$ as function of the masses M and m_i.

c) The decaying particle is now considered in a frame Σ' in which the particle has the velocity β along the x^3 axis $(c = 1)$. What is the relation between energies and angles in Σ' with the respective quantities in Σ?

d) For the special case $m_1 = m_2 = 0$ (e.g. decay into two photons) determine the angle θ' between the directions of flight of the decay products in Σ' (i.e. the angle between $\mathbf{p}_1'$ and $\mathbf{p}_2'$). What are the extremal values of θ'? In particular, discuss the cases $\beta = 0$ and $\beta \to 1$.