Exercise 2.1 *Classical, relativistic, free point particle* (2 points)

We consider a classical, free point particle in special relativity and aim at describing it within the Lagrangian and Hamiltonian formalisms. The particle moves through space on some trajectory $\vec{x}(t) = (x^k(t))$ in an inertial frame Σ. We demand that the motion is derived from Hamilton's action principle and that the action S corresponding to the particle is Lorentz invariant. The only Lorentz-invariant quantity for some motion between the points $\vec{a}_1 = \vec{x}(t_1)$ and $\vec{a}_2 = \vec{x}(t_2)$ and fixed times t_1 and t_2 is given by the integral $\int_{\vec{a}_1}^{\vec{a}_2} ds$ with $ds^2 = c^2dt^2 - d\vec{x}^2$ the Lorentz-invariant line element.
(In this exercise it is instructive to keep $c \neq 1$.)

a) The above considerations motivate the ansatz

$$S = k \int_{(t_1,\vec{a}_1)}^{(t_2,\vec{a}_2)} ds$$

for the action, with k some preliminary constant. What is the Lagrangian function L corresponding to S? Determine the value of k by comparing L with the non-relativistic Lagrangian function $L_{\mathrm{non-relat}} = \frac{1}{2}m\dot{\vec{x}}^2$.

[Result: $L = -mc^2/\gamma$ with $\gamma = 1/\sqrt{1 - v^2/c^2}$, $v = |\dot{\vec{x}}|$.]

b) Determine the relativistic three-momentum $\vec{p} = (p^k) = \dfrac{\partial L}{\partial \dot{\vec{x}}}$ and the Lagrangian equations of motion.

c) Derive the Hamilton function

$$H(x^k, p^k) = \sum_{l=1}^{3} \dot{x}^l p^l - L(x^k, \dot{x}^k).$$

d) The four-velocity u^μ of the particle is defined by $u^\mu = (\gamma c, \gamma\dot{\vec{x}})$. Check that the space-like components of the momentum four-vector, defined as $p^\mu = mu^\mu$, indeed coincide with $\vec{p}$ as defined above. What is the relation between the time-like component p^0 and the Hamilton function H?

e) Calculate the invariant square p^2 of p^μ. What is the meaning and non-relativistic limit of the resulting relation?

Please turn over!

Exercise 2.2 *Classical, relativistic four-momentum conservation* (1 point)

Consider a system of N classical point particles, which may interact with each other, but not with any external particles or fields. The Lagrangian function $L(\vec{x}_i, \dot{\vec{x}}_i, t)$ is, thus, invariant under space and time translations,

$$L(\vec{x}_i, \dot{\vec{x}}_i, t) = L(\vec{x}_i + \vec{a}, \dot{\vec{x}}_i, t + a^0),$$

where a^μ is an arbitrary constant four-vector describing the translation.

a) Derive the conservation of the sum of relativistic three-momenta $\vec{p}_i = \dfrac{\partial L}{\partial \dot{\vec{x}}_i}$ from the relation $\dfrac{\partial L}{\partial \vec{a}} = 0$.

b) Define the Hamilton function H as the total relativistic energy of the system and show that it is conserved owing to the relation $\dfrac{\partial L}{\partial a^0} = 0$.

Exercise 2.3 *Mandelstam variables for* $2 \to 2$ *particle reactions* (2 points)

We consider the particle reaction $A(p_1) + B(p_2) \to C(p_3) + D(p_4)$ with the four-momenta p_i and $p_i^2 = m_i^2$, where m_i are the particle masses. The Mandelstam variables s, t, u are defined by $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$.

a) Show that $s + t + u = \sum_{i=1}^4 m_i^2$.

b) In the centre-of-mass frame Σ the momenta are given by

$$p_{1,2}^\mu = (E_{1,2}, 0, 0, \pm q), \qquad p_{3,4}^\mu = (E_{3,4}, \pm k \cos\phi \sin\theta, \pm k \sin\phi \sin\theta, \pm k \cos\theta).$$

Which constraints result from four-momentum conservation?

c) Calculate s, t, u in the system Σ. What is the meaning of the variable s? Express t and u in terms of s, the angles θ, ϕ, and the masses m_i.

d) Consider the special case $m_1 = m_3 = 0$, $m_2 = m_4 = m$ in the rest frame Σ' of particle B (e.g. Compton scattering at an electron at rest). Derive the relation between E_3' and θ_3' for given energy E_1' upon evaluating $t = (p_1 - p_3)^2 = (p_2 - p_4)^2$ in two different ways and using energy conservation.