Exercise 3.1 *Green function of Schrödinger's equation* (1 point)

The *Green function* G of Schrödinger's equation of a particle of mass m in the potential V is defined via the condition

$$\left(i\frac{\partial}{\partial t}+\frac{1}{2m}\nabla_x^2-V(\mathbf{x})\right)G(t,\mathbf{x};t',\mathbf{x}')=\delta(t-t')\delta^{(3)}(\mathbf{x}-\mathbf{x}').$$

a) Determine the Fourier transform of the Green function G_0 of the *free* Schrödinger equation with $V(\mathbf{x})=0$, i.e. write G_0 in the form

$$G_0(t,\mathbf{x};t',\mathbf{x}')=\frac{1}{(2\pi)^4}\int \mathrm{d}\omega\int \mathrm{d}^3p\, e^{-i\omega(t-t')}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\tilde{G}_0(\omega,\mathbf{p})$$

and determine the function $\tilde{G}_0(\omega,\mathbf{p})$. Why can the free Green function only depend on $t-t'$ and $\mathbf{x}-\mathbf{x}'$?

b) Show that the Green function of the full Schrödinger equation with $V\neq 0$ fulfills the equation

$$G(t,\mathbf{x};t',\mathbf{x}')=G_0(t,\mathbf{x};t',\mathbf{x}')+\int \mathrm{d}\tilde{t}\,\mathrm{d}^3\tilde{x}\;G_0(t,\mathbf{x};\tilde{t},\tilde{\mathbf{x}})V(\tilde{\mathbf{x}})G(\tilde{t},\tilde{\mathbf{x}};t',\mathbf{x}').$$

Exercise 3.2 *Hamiltonian of the classical, free Klein-Gordon field* (1.5 points)

Consider the Lagrangian $\mathcal{L}=(\partial\phi)(\partial\phi)^*-m^2\phi\phi^*$ of the free, complex Klein-Gordon field ϕ.

a) Derive the corresponding Hamiltonian $\mathcal{H}=\pi\dot{\phi}+\pi^*\dot{\phi}^*-\mathcal{L}$, with $\pi=\dfrac{\partial\mathcal{L}}{\partial\dot{\phi}}$ denoting the canonical conjugate field to ϕ.

b) Using the plane-wave solution $\phi=\int \mathrm{d}\tilde{p}\,\left[a(\mathbf{p})\mathrm{e}^{-ipx}+b^*(\mathbf{p})\mathrm{e}^{+ipx}\right]$ with some (square-integrable) arbitrary complex functions $a(\mathbf{p})$, $b(\mathbf{p})$ show that

$$H=\int \mathrm{d}^3x\,\mathcal{H}=\int \mathrm{d}\tilde{p}\,p_0\,\left[|a(\mathbf{p})|^2+|b(\mathbf{p})|^2\right],$$

where the momentum-space integral is defined by $\displaystyle\int \mathrm{d}\tilde{p}\equiv\int\left.\frac{\mathrm{d}^3p}{(2\pi)^3 2p_0}\right|_{p_0=\sqrt{\mathbf{p}^2+m^2}}$.

c) Employing the (time-independent) scalar product

$$(\phi,\chi)\equiv i\int \mathrm{d}^3x\;\left[\phi(x)^*(\partial_0\chi(x))-\chi(x)(\partial_0\phi(x)^*)\right]$$

of two free Klein-Gordon fields ϕ,χ, is it possible to interpret H as expectation value of the energy operator $P_0=i\partial_0$ of the field modes, i.e. as $(\phi,P_0\phi)$?

Please turn over!

Exercise 3.3 *Gauge invariance and minimal substitution* (1 point)

a) The electric and magnetic fields $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi$ and $\mathbf{B} = \nabla \times \mathbf{A}$ are invariant under the *gauge transformation* of the scalar potentials $\phi \to \phi + \frac{\partial \omega}{\partial t}$ and of the vector potential $\mathbf{A} \to \mathbf{A} - \nabla\,\omega$, where $\omega(t, \mathbf{x})$ is an arbitrary function of space and time.

Show that Schrödinger's equation of a spinless particle with charge q in the electromagnetic field,

$$\left(i\frac{\partial}{\partial t} + \frac{1}{2m}\left(\nabla_x - iq\mathbf{A}(t,\mathbf{x})\right)^2 - q\phi(t,\mathbf{x}) \right)\psi(t,\mathbf{x}) = 0,$$

is invariant under gauge transformations if the following transformation of the wave function ψ is applied simultaneously:

$$\psi(t,\mathbf{x}) \to e^{-iq\omega(t,\mathbf{x})}\psi(t,\mathbf{x})\,.$$

Comment: The introduction of the interaction with the electromagnetic field by the replacements $\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + iq\phi$ and $\nabla \to \nabla - iq\mathbf{A}$ is also called *minimal substitution*.

b) Carrying out the minimal substitution in the free Klein-Gordon equation of a scalar field with charge q we obtain the Klein-Gordon equation with the electromagnetic interaction:

$$\left[(\partial_\mu + iqA_\mu)(\partial^\mu + iqA^\mu) + m^2\right]\Phi(x) = 0$$

with $A^\mu = (\phi, \mathbf{A})$. Show in analogy to Schrödinger's equation that this equation is invariant under gauge transformations if the following transformation of the scalar field is applied:

$$\Phi'(x) \to e^{-iq\omega(x)}\Phi(x)\,.$$