Exercise 5.1 *Momentum of the quantized free scalar field* (1.5 points)

Energy P^0 and momentum $\mathbf{P}$ of a free, real scalar field $\phi(x)$ are determined by the energy-momentum tensor via

$$P^0 = \int d^3\mathbf{y}\, \frac{1}{2} \left\{ \pi(y)^2 + [\nabla\phi(y)]^2 + m^2\phi(y)^2 \right\}, \qquad \mathbf{P} = -\int d^3\mathbf{y}\, \pi(y)[\nabla\phi(y)],$$

where $\pi(x) = \dot{\phi}(x)$ is the canonical conjugate operator to the field operator $\phi(x)$.

a) Using the canonical commutator relations show

$$[\phi(x), P^\mu] = i\partial^\mu\phi(x).$$

b) How does P^μ act on the one-particle wave function $\varphi_{\mathbf{p}}(t,\mathbf{x}) = \langle 0|\phi(t,\mathbf{x})|\mathbf{p}\rangle$ with fixed momentum $\mathbf{p}$, where the action of an operator A on $\varphi_{\mathbf{p}}(t,\mathbf{x})$ is defined by $A\varphi_{\mathbf{p}}(t,\mathbf{x}) = \langle 0|\phi(t,\mathbf{x})A|\mathbf{p}\rangle$.

c) Employing a), derive the following identity for translations by a constant 4-vector a^μ:

$$\exp\{ia_\mu P^\mu\}\,\phi(x)\,\exp\{-ia_\nu P^\nu\} = \phi(x+a).$$

Exercise 5.2 *Identities of the scalar field operator* (1 point)

Consider the field operator $\phi(x)$ of the free, real Klein-Gordon field.

a) Show that

$$[\phi(x), \phi(y)] = \int d\tilde{k}\, \left[\mathrm{e}^{-ik(x-y)} - \mathrm{e}^{+ik(x-y)}\right]$$

and argue why $[\phi(x), \phi(y)] = 0$ for $(x-y)^2 < 0$, as demanded by causality.

b) Proof the relation between time ordering and normal ordering:

$$T\left[\phi(x)\phi(y)\right] = :\phi(x)\phi(y): + \langle 0\,|T\left[\phi(x)\phi(y)\right]|\,0\rangle.$$

Please turn over!

Exercise 5.3 *Charge operator of the free, complex scalar field* (0.5 points)

Consider the field operator $\phi(x)$ of the free, complex Klein-Gordon field describing a spin-0 boson of mass m and electric charge q. The plane-wave expansion of $\phi(x)$ is given by

$$\phi(x) = \int d\tilde{p}\ \left[a(\mathbf{p})\mathrm{e}^{-ipx} + b^\dagger(\mathbf{p})\mathrm{e}^{ipx}\right],$$

where $a(\mathbf{p})$, $a^\dagger(\mathbf{p})$ are the annihilation and creation operators for the particle, respectively, and likewise $b(\mathbf{p})$, $b^\dagger(\mathbf{p})$ for the corresponding antiparticle. Express the charge operator

$$Q = \int d^3\mathbf{x}\ iq\ :\left[\phi^\dagger(\partial_0\phi) - (\partial_0\phi)^\dagger\phi\right]:$$

in terms of the annihilation and creation operators of the momentum eigenstates.