**Exercise 6.1** *Normalization of multi-particle states* (0.5 points)

Show that the $n$-particle states in (bosonic) Fock space,

$$|\mathbf{p}_1, \ldots \mathbf{p}_n\rangle = a^\dagger(\mathbf{p}_1) a^\dagger(\mathbf{p}_2) \ldots a^\dagger(\mathbf{p}_n)|0\rangle,$$

are normalized according to

$$\langle \mathbf{p}_1, \ldots \mathbf{p}_n | \mathbf{k}_1, \ldots, \mathbf{k}_m\rangle = \delta_{mn}\,(2\pi)^{3n} \sum_P \prod_i (2p_i^0)\,\delta^3(\mathbf{p}_i - \mathbf{k}_{P(i)}),$$

where the sum is over all permutations $P$ of the indices $(1 \ldots n)$.

**Exercise 6.2** *Wick-Theorem for bosonic fields* (2 points)

The purpose of this exercise is to prove Wick's theorem for bosonic, real, free field operators $\phi_i \equiv \phi_i(x_i)$, which states

$$\begin{aligned} T[\phi_1 \cdots \phi_n] \;=\; & :\phi_1 \cdots \phi_n: \; + \sum_{\text{pairs } ij} :\wick{\phi_1 \cdots \c{\phi}_i \cdots \c{\phi}_j \cdots \phi_n}: \\ & + \sum_{\text{double pairs } ij,kl} :\wick{\phi_1 \cdots \c1{\phi}_i \cdots \c2{\phi}_k \cdots \c1{\phi}_j \cdots \c2{\phi}_l \cdots \phi_n}: \; + \;\ldots \end{aligned}$$

with the contractions representing propagators, $\wick{\c{\phi}_i \c{\phi}_j} = \langle 0|T[\phi_i\phi_j]|0\rangle$. For $n = 2$ the theorem has already been proven in Exercise 5.2. We organize the general proof in terms of two steps. Without losing generality we can assume in the proof that $t_n = x_n^0$ is the smallest time variable, i.e. $T[\phi_1 \cdots \phi_n] = T[\phi_1 \cdots \phi_{n-1}]\phi_n$.

a) First prove the lemma

$$:\phi_1 \cdots \phi_{n-1}: \phi_n \;=\; :\phi_1 \cdots \phi_n: \; + \sum_{k=1}^{n-1} :\wick{\phi_1 \cdots \c{\phi}_k \cdots \c{\phi}_n}: \;.$$

To this end, split $\phi_n$ according to $\phi_n = \phi_n^{(+)} + \phi_n^{(-)}$ into their positive and negavite frequency parts $\phi_n^{(\pm)}$, i.e. $\phi_n^{(+)}$ involves only annihilation operators and $\phi_n^{(-)}$ only creation operators. For $\phi_n^{(+)}$ the lemma is trivially verified, for $\phi_n^{(-)}$ you can proceed via induction in $n$.

b) Argue that the lemma of a) trivially generalizes to cases where contractions already appear inside the normal orderings, for example:

$$\begin{aligned} :\wick{\phi_1 \cdots \c{\phi}_i \cdots \c{\phi}_j \cdots \phi_{n-1}}: \phi_n \;=\; & :\wick{\phi_1 \cdots \c{\phi}_i \cdots \c{\phi}_j \cdots \phi_n}: \\ & + \sum_{k=1}^{n-1} :\wick{\phi_1 \cdots \c1{\phi}_i \cdots \c2{\phi}_k \cdots \c1{\phi}_j \cdots \c2{\phi}_n}\;. \end{aligned}$$

c) Prove Wick's theorem via induction in $n$ using the result of b).