Exercise 10.1 *Interaction picture and time-dependent perturbation theory* (3 points)

We consider a quantum-mechanical system that is described by an "unperturbed" Hamiltonian $\hat{H}_0$, which is time independent, and a time-dependent perturbation $\hat{H}'(t)$. The eigenstates $|\phi_n\rangle$ and corresponding eigenvalues E_n of $\hat{H}_0$ are supposed to be known.

a) A Schrödinger state $|\psi(t)\rangle$ at any time t can be expressed in terms of a linear combination of stationary states $|\phi_n(t)\rangle = \exp\{-\mathrm{i}E_n(t-t_0)/\hbar\}|\phi_n\rangle$ of the unperturbed Hamiltonian $\hat{H}_0$, i.e.

$$|\psi(t)\rangle \;=\; \sum_n c_n(t)\,|\phi_n(t)\rangle,$$

with time-dependent coefficients $c_n(t)$ and t_0 being a fixed reference time. Express the corresponding state $|\psi(t)\rangle_I$ in the interaction picture in terms of the coefficients $c_n(t)$ and the states $|\phi_n\rangle$.

b) Derive a set of first-order differential equations for the time evolution of all $c_n(t)$ from Schrödinger's equation, in which matrix elements $H'_{nm}(t) = \langle\phi_n|\hat{H}'(t)|\phi_m\rangle$ of the perturbation are the only unknown quantitites apart from the $c_n(t)$'s.

c) Assuming $H'_{nm}(t)$ as small quantities, describe an iterative way to solve the equations for $c_n(t)$ with the starting condition $c_n(t_0) = \delta_{nj}$.

Exercise 10.2 *Time-dependent perturbation theory to second order* (2 points)

We consider the same system as in the previous exercise, using the same definitions of $\hat{H}_0$, $\hat{H}'(t)$, $|\phi_n\rangle$, and E_n. The transition probability for a state $|\psi(t)\rangle$ with $|\psi(t_0)\rangle = |\phi_i\rangle$ for a given time t_0 to a state $|\phi_f\rangle$ at some later time t is given by

$$W_{if} \;=\; |\langle\phi_f|U_I(t,t_0)|\phi_i\rangle|^2,$$

where

$$U_I(t,t_0) \;=\; \mathrm{T}\exp\left\{-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\mathrm{d}t'\,\hat{H}'_I(t')\right\}$$

is the time-evolution operator in the interaction picture, represented by a time-ordered exponential function, and $\hat{H}'_I(t)$ is the perturbation in the interaction picture.

a) Show that the probability W_{ii} that the system is found at time t in the same state $|\phi_i\rangle$ as at time t_0 is given by

$$W_{ii} \;=\; 1+\frac{1}{\hbar^2}\left(\int_{t_0}^{t}\mathrm{d}t_1\,\langle\phi_i|\hat{H}'_I(t_1)|\phi_i\rangle\right)^2-\frac{1}{\hbar^2}\int_{t_0}^{t}\mathrm{d}t_1\int_{t_0}^{t}\mathrm{d}t_2\,\langle\phi_i|\hat{H}'_I(t_1)\hat{H}'_I(t_2)|\phi_i\rangle+\ldots,$$

which is valid up to quadratic order in the perturbation.

b) Calculate W_{if} for $f \neq i$ up to second order as well and prove the unitarity relation $\sum_f W_{if} = 1$ upon using the result for W_{ii} from a).

Please turn over!

Exercise 10.3 *Wave-like perturbation of hydrogen ground state* (3 points)

The ground state ϕ_0 of a hydrogen atom is subjected to a time-dependent perturbation of the form

$$\hat{H}'(t) \;=\; A\cos(k\hat{x}_3 - \omega t),$$

where A, k, and ω are real positive constants. Our aim is to approximately calculate the transition rate of the atom to an ionized state in which the electron is emitted with a momentum $\vec{p}$. We will approximate the (stationary) final state of the electron by a free-particle wave function $\psi_{\vec{p}}(\vec{x}) \propto \exp\{-\mathrm{i}\vec{p}\cdot\vec{x}/\hbar\}$. We neglect spin effects.

a) Since $\psi_{\vec{p}}$ is not normalizable in full space, we temporarily consider the electron confined to a cube of edge length L with periodic boundary conditions, where L is much larger than Bohr's atomic radius. Calculate the density $\rho(E_p)$ of free-particle states in the cube, where $E_p = \vec{p}^{\,2}/(2m)$ is the energy of the free electron, and normalize $\psi_{\vec{p}}$ to the volume of the cube.

b) Calculate the transition matrix element $\langle\psi_{\vec{p}}|\mathrm{e}^{ik\hat{x}_3}|\phi_0\rangle$ in the asymptotic limit $L \to \infty$.

c) Using the general result derived in the lecture for periodic perturbations to first order, derive the transition rate $\mathrm{d}R$ for an electron that is emitted into the direction of $\vec{p}$ within the solid angle $\mathrm{d}\Omega$. Discuss the angular dependence of the rate.