Exercise 8.1 *Irreducible spherical tensors* (2 points)

a) Form an irreducible spherical tensor $T_m^{(3)}$ out of products $u_a v_b w_c$ of the components u_a, v_b, w_c of the three cartesian vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$.

b) Prove that

$$Z_m^{(j)} = \sum_{m_1,m_2} X_{m_1}^{(j_1)} Y_{m_2}^{(j_2)} \langle j_1 j_2 m_1 m_2 | jm \rangle \tag{1}$$

is an irreducible spherical tensor operator of rank j if $X^{(j_1)}$ and $Y^{(j_2)}$ are both irreducible spherical tensors of ranks j_1 and j_2, respectively.

Exercise 8.2 *WKB method for s-states in central potentials* (2 points)

Consider a particle of mass m in a central potential $V(r)$ in 3 dimensions ($r = |\vec{x}|$). For vanishing angular momentum ($l = 0$), the wave function $\psi(\vec{x})$ is spherically symmetric and given by $\psi(\vec{x}) \propto u(r)/r$, where the radial function $u(r)$ plays the role of the wave function of the equivalent 1-dimensional problem with an effective potential $V(r)$ (no centrifugal term for $l = 0$).

a) In order to find an approximation for the energy eigenvalues of bound states, partition the r-range in different regions with appropriate boundary or matching conditions for $u(r)$.

b) Following the strategy of the lecture, construct the WKB approximation for $u(r)$ in the full r-range. As resulting condition on the nth energy eigenvalue E_n you should obtain

$$\oint \mathrm{d}r\, p_r(r) \;=\; h\left(n + \frac{3}{4}\right), \qquad n = 0, 1, 2, \ldots,$$

where h is Planck's constant and p_r the radial momentum.

Please turn over!

Exercise 8.3 *Linear potential and WKB method* (2 points)

Consider a particle with mass m in a one-dimensional potential $V(x) = \varepsilon|x|$ with $\varepsilon > 0$.

a) Determine an approximation for the energy eigenvalues E_n ($n = 0, 1, 2, \dots$) using the WKB method.

b) Derive the antisymmetric wave functions $\psi(x) = -\psi(-x)$ upon using the results of Exercise 2.3 with the help of a symmetry argument. To which n-values of a) do these wave functions correspond? Compare the exact energy eigenvalues E_n with the respective approximations obtained in a) numerically.

   (Hint: Take the zeroes of the Airy function from the literature.)

Exercise 8.4 *Second-order perturbation theory – a delicate case* (2 points)

Consider a three-state system with the following Hamiltonian in matrix representation,

$$\hat{H} = \begin{pmatrix} E_1 & 0 & a \\ 0 & E_1 & b \\ a^* & b^* & E_2 \end{pmatrix},$$

where a, b are considered as small perturbations ($|a|, |b| \ll |E_2 - E_1|$) and $E_{1,2}$ are the (real) energy eigenvalues of the unperturbed system.

a) Calculate the exact energy eigenvalues of the system and expand them in the small quantities a, b to the first non-trivial order.

b) Calculate the energy eigenvalues using second-order perturbation theory. Alternatively, employ second-order perturbation theory ignoring the issue of degeneracy and compare the two results with result of a).