Advanced Quantum Mechanics

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Revision questions: just to give some orientation and to serve as a start for discussions in the next tutorial

—**Not** *covering all you should know*—

1. What is the statement of Wigner's theorem?

2. What is the difference between a group and a representation thereof? Can we always reconstruct the group from a given representation?

3. A certain representation $\mathcal{U}$ of a group $\mathcal{G}$ on a vector space, assigns matrices to the elements $g \in \mathcal{G}$ with the following form

$$\mathcal{U}(g) = \left(\begin{array}{c:c} \mathcal{A}(g) & \mathcal{B}(g) \\ \hdashline \mathbf{0}_{k\times n} & \mathcal{C}(g) \end{array}\right), \tag{1}$$

where the gray blocks correspond to non-zero entries. Is there an invariant subspace? Is this representation reducible or irreducible? How would the matrices look like if the representation were completely reducible? The Hamiltonian $\hat{H}$ is invariant under the action of $\mathcal{G}$, and we know that the irreducible representations of the group are of dimension $d > 1$, what can we say about the energy spectrum of $\hat{H}$?

4. Why are group representations important in quantum mechanics? Why is irreducibility important? What can we infer from knowing all representations of a symmetry group of quantum mechanical system?

5. What happens to irreducible representations when enlarging or reducing the symmetry?

6. What is the statement of Schur's lemma and its "inverse"?

7. Prove that all abelian groups have only 1-dimensional irreducible representations.

8. Prove that if a physical system is invariant under a certain symmetry transformation, generated by the operator $\hat{A}$, then $[\hat{H}, \hat{A}] = 0$, where $\hat{H}$ is the Hamiltonian of the system.

9. What is the relation between a Lie group and a Lie algebra? Give an example.

10. What is the statement of Bloch's theorem? What are Bloch waves?

11. What are Wigner's D and d functions?

12. What is the relation between $SO(3)$ and $SU(2)$?

13. Given the action of a group, for example the rotation group $SO(3)$, in Euclidean space $\mathbb{R}^3$, how is the action of the group on a wave function $\Psi \in \mathcal{L}^2(\mathbb{R}^3)$ naturally defined?

14. What is the definition of an angular momentum $\vec{J}$ in quantum mechanics? If we write the eigenvalues of $\vec{J}^2$ as $\hbar^2 j(j+1)$, which conditions must j fulfill?

15. The orbital angular momentum $\vec{L}$ and the spin angular momentum $\vec{S}$ are the generators of which symmetry transformations? Can you write the general form of the corresponding unitary operators? How are these transformations realized in the Hilbert space of spinor wave functions?

16. Let $\vec{J} = \vec{J}^{(1)} + \vec{J}^{(2)}$ be the sum of two angular momenta. What are the possible eigenvalues of $\vec{J}^2$? What can you say about the following commutators?:

$$[\vec{J}^2, J_x], \qquad [\vec{J}^2, (\vec{J}^{(1)})^2], \qquad [\vec{J}^2, \vec{J}^{(1)} \cdot \vec{J}^{(2)}], \qquad [\vec{J}^2, J_z^{(1)}]. \tag{2}$$

17. Can you solve the general eigenvalue problem of angular momentum without any additional resources?

18. Can you add two quantum mechanical angular momenta without any additional resources?

19. How can operators be classified according to their behaviour under rotations? What are irreducible spherical tensors?

20. What are Clebsch–Gordan coefficients? What is the Clebsch–Gordan series?

21. What does the Wigner–Eckart theorem imply for scalar and vector operators, when working in a subspace of dimension $(2j + 1)$ of fixed angular momentum j?

22. Where is the WKB method applicable? How does it work?

23. How does Ritz's variational method work for excited states?

24. How does time-independent perturbation theory work? What are its assumptions? Are there subtleties?

25. What are the differences between Schrödinger, Heisenberg, and interaction picture?

26. Using first-order perturbation theory for a system with the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}'(t)$, we found that the transition probability from the state $|\varphi_i\rangle$ to the state $|\varphi_{f\neq i}\rangle$ is given by

$$W_{fi}(t, t_0) = \frac{1}{\hbar^2} \left| \int_{t_0}^{t} dt'\, e^{\mathrm{i}\omega_{fi}t'} \langle\varphi_f|\hat{H}'(t')|\varphi_i\rangle \right|^2. \tag{3}$$

Explain all the elements in this expression.

27. Let us consider the transition rate

$$\frac{W_{fi}}{T} \underset{T\to+\infty}{\widetilde{\quad}} \frac{2\pi}{\hbar}|h_{fi}|^2\delta(E_f - E_i - \hbar\omega) + \frac{2\pi}{\hbar}|h_{if}|^2\delta(E_f - E_i + \hbar\omega). \tag{4}$$

Explain the meaning of all the terms and describe for which transitions and under which conditions this rule can be used. What is Fermi's golden rule?

28. What is the Hamiltonian of a charged particle (with and without spin) in an electromagnetic field?

29. How are Green's functions defined for the time(-in)dependent Schrödinger equation? Suppose you know an orthonormal basis of $\hat{H}$. How can you get the Green's functions?

30. Derive the Born series from the Lippman–Schwinger equation.

31. You want to calculate the elastic scattering cross section σ of a short-range central potential at low energies: which method would you use and why? What if you aim for the high-energy behaviour of σ? Does your answer depend upon the strength of the potential?

32. The scattering amplitude in first Born approximation reads

$$f_k(\Omega) = -\frac{M}{2\pi\hbar^2} \int \mathrm{d}^3x\, e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}\, V(\vec{x}). \tag{5}$$

Describe all quantities in this expression as well as the approximations made in its derivation.

33. What is the statement of the optical theorem? Can it be understood intuitively?