Exercises to Group Theory for Physicists — Sheet 1
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Exercise 1.1 *Symmetry-induced degeneracy* (2 points)

Consider a quantum-mechanical system with Hamiltonian $\hat{H}$ which has an orthonormal basis $\{|n\rangle\}_{n=1}^{N}$ of energy eigenstates, i.e. $\hat{H}\,|n\rangle = E_n\,|n\rangle$. The case $N = \infty$ is possible.

Show that the existence of degenerate energy eigenstates, i.e. $E_n = E_m$ for some $n \neq m$, can only be enforced by (at least) two symmetry operators U_1, U_2 if $[U_1, U_2] \neq 0$. In other words, a *non-abelian* group of symmetry operators is required for $\hat{H}$ to possess degenerate energy eigenstates as a consequence of symmetries.

Exercise 1.2 *Some basic facts about groups* (3 points)

a) Given two different group elements f and g, show that fg and gf are in the same equivalence class.

b) Show that every group of even order has an element g, $g \neq e$, with $g^2 = e$, where e is the identity element.

c) The centre $Z(G)$ of a group G is the set of all elements $g \in G$ that commute with all other elements of G. Show that $Z(G)$ is a normal, abelian subgroup of G.

Exercise 1.3 *Simplified version of Wigner's theorem* (2 points)

a) Show that a linear operator U that preserves all norms of states in a Hilbert space, i.e.

$$\|\psi\| = \|U\psi\| \qquad \text{for all} \qquad |\psi\rangle \in \mathcal{H}, \tag{1}$$

is unitary.

b) Analogously, show that an antilinear operator U that satisfies the condition (1), where antilinearity means

$$U\Big(a|\psi\rangle + b|\phi\rangle\Big) = a^* U|\psi\rangle + b^* U|\phi\rangle, \qquad |\psi\rangle, |\phi\rangle \in \mathcal{H}, \quad a, b \in \mathbb{C}, \tag{2}$$

is antiunitary, i.e. $\langle U\psi|U\phi\rangle = \langle\psi|\phi\rangle^*$. Note that the adjoint of an antiunitary operator U is defined by $\langle U\phi|\psi\rangle = \langle\phi|U^\dagger\psi\rangle^*$.

Comment: Wigner more generally showed that the requirement $|\langle U\phi|U\psi\rangle| = |\langle\phi|\psi\rangle|$ for all $|\phi\rangle, |\psi\rangle$ implies that U is either unitary (which implies linearity) or antiunitary (which implies antilinearity). A complete proof can be found in S. Weinberg, *The Quantum Theory of Fields*, Vol. I, p. 91.

Please turn over!

Exercise 1.4 *Baker-Campbell-Hausdorff formula – special case* (2 points)

Consider the (not necessarily commuting) operators A and B defined on some Hilbert space. The exponential function e^A of an operator A is defined via its power series, which can be assumed to converge in the following.

a) Proof $\frac{\mathrm{d}}{\mathrm{d}\alpha}\,\mathrm{e}^{\alpha A} = A\,\mathrm{e}^{\alpha A}$ (with $\alpha \in \mathbb{R}$) and $(\mathrm{e}^A)^{-1} = \mathrm{e}^{-A}$.

b) Proof the following special cases of the BCH formula,

$$\mathrm{e}^A\,\mathrm{e}^B = \mathrm{e}^B\,\mathrm{e}^A\,\mathrm{e}^{[A,B]}, \qquad \mathrm{e}^{A+B} = \mathrm{e}^A\,\mathrm{e}^B\,\mathrm{e}^{-[A,B]/2},$$

which are valid if $[A,[A,B]] = [B,[B,A]] = 0$.