Exercises to Group Theory for Physicists — Sheet 2
Prof. S. Dittmaier and Dr. P. Maierhöfer, Universität Freiburg, SS19

Exercise 2.1 *Galilei transformations of a qm. one-particle system* (6 points)

The group of genuine Galilei transformations (without rotations and translations) is a 3-dimensional Lie group generated by the operators

$$\vec{G} = \frac{1}{\hbar}\left(\hat{\vec{p}}t - m\hat{\vec{x}}\right). \tag{1}$$

For the system of a point particle, $\hat{\vec{x}}$ and $\hat{\vec{p}}$ are the usual position and momentum operators. The roles of the parameters m and t will become clear in the following. The unitary symmetry operators $U(\vec{v})$ acting on the one-particle Hilbert space of states $|\psi\rangle$ are given by

$$U(\vec{v}) = \exp\left\{-\mathrm{i}\vec{G}\cdot\vec{v}\right\} \tag{2}$$

with the velocity $\vec{v}$, defining boosts connecting two different frames of inertia.

a) Calculate the commutators $[\hat{x}_a, G_b]$, $[\hat{p}_a, G_b]$, and $[G_a, G_b]$. Is the group generated by $\vec{G}$ abelian or nonabelian?

b) Show that $U(\vec{v})$ can be written in the form

$$U(\vec{v}) = \exp\left\{\frac{\mathrm{i}}{\hbar}m\hat{\vec{x}}\cdot\vec{v}\right\}\exp\left\{-\frac{\mathrm{i}}{\hbar}t\hat{\vec{p}}\cdot\vec{v}\right\}\mathrm{e}^{\mathrm{i}\chi(v)}, \tag{3}$$

with some real function $\chi(v)$, and determine the function $\chi(v)$ explicitly.

c) Calculate $[\hat{p}_a, U(\vec{v})]$ and show that the state $U(\vec{v})|\vec{p}\rangle$ is a momentum eigenstate if $|\vec{p}\rangle$ is a momentum eigenstate with momentum $\vec{p}$. What is the role of m?

d) Determine the phase factor in the relation $|\vec{p}\,'\rangle = \mathrm{e}^{-\mathrm{i}\phi(\vec{p},\vec{v})}\,U(\vec{v})|\vec{p}\rangle$, where the momentum eigenstates are normalized according to $\langle\vec{x}|\vec{p}\rangle = \mathrm{e}^{\mathrm{i}\vec{p}\cdot\vec{x}/\hbar}$ as usual.

e) Calculate $[\hat{x}_a, U(\vec{v})]$ and show that the state $U(\vec{v})|\vec{x}\rangle$ is a position eigenstate if $|\vec{x}\rangle$ is a position eigenstate with position $\vec{x}$. What is the role of t?

f) Given a state $|\psi\rangle$ with wave function $\psi(\vec{x}) = \langle\vec{x}|\psi\rangle$, determine the wave function $\psi'(\vec{x}) = \langle\vec{x}|\psi'\rangle$ for the transformed state $|\psi'\rangle = U(\vec{v})|\psi\rangle$ in terms of $\psi(\vec{x})$.

Please turn over!

Exercise 2.2 *Discrete rotations in two dimensions – the cyclic groups* (3 points)

For a given natural number n, consider the group of discrete rotations about integer multiples of the angle $\frac{2\pi}{n}$ in two dimensions, which are represented by the matrices

$$R(\phi_k) = \begin{pmatrix} \cos\phi_k & -\sin\phi_k \\ \sin\phi_k & \cos\phi_k \end{pmatrix}, \qquad \phi_k = \frac{2\pi k}{n} \qquad k = 0, 1, \ldots, n-1. \tag{4}$$

These matrices define a two-dimensional representation of the *cyclic group* C_n.

a) Determine the similarity transformation that reduces the representation (4) to two irreducible representations.

b) The regular representation of C_3 (and analogously for C_n) is defined by the following three matrices:

$$D(e) = \mathbb{1}, \qquad D(g) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad D(g^2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{5}$$

where g is the *generating element* of the group. Similarly to a) fully reduce this representation.

Hint: Introduce $\epsilon = \mathrm{e}^{2\pi\mathrm{i}/3}$, with $\epsilon^2 = \epsilon^*$, $1 + \epsilon + \epsilon^2 = 0$.

c) C_n possesses n one-dimensional inequivalent representations. Guess them from the pattern observed for C_3 in b).