**Exercises to Group Theory for Physicists — Sheet 8**
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**Exercise 8.1** *The relation between U(N) and SU(N)* (2 points)

a) Find the centre of U(N), i.e. the elements that commute with all elements of the group. The centre constitutes a subgroup of U(N). What is this group?

b) Which elements does the centre of U(N) have in common with the SU(N) subgroup, and what group do these elements constitute? Use this to build a group from the subgroups that is isomorphic to U(N).

**Exercise 8.2** *Some su(3) relations* (4 points)

In the 3-dimensional defining representation of SU(3) the generators $T^a = \lambda^a/2$ are given by the Gell-Mann matrices $\lambda^a$ ($a = 1, \dots, 8$), which obey the relations

$$[\lambda^a, \lambda^b] = 2\mathrm{i} f^{abc}\lambda^c, \qquad \lambda^a = (\lambda^a)^\dagger, \qquad \mathrm{Tr}(\lambda^a) = 0, \qquad \mathrm{Tr}(\lambda^a\lambda^b) = 2\delta^{ab}, \tag{1}$$

with $f^{abc}$ being the structure constants of su(3).

a) Calculate the matrices $A = \sum_{a=1}^{3}(\lambda^a)^2$ and $B = \sum_{a=1}^{8}(\lambda^a)^2$. Are $A$ and $B$ Casimir operators?

b) Explain why the anticommutator of $\lambda^a$ can be written as $\{\lambda^a, \lambda^b\} = C\delta^{ab}\mathbb{1} + 2d^{abc}\lambda^c$ with the real constants $C$ and $d^{abc}$. Determine $C$.

c) Prove

$$\mathrm{Tr}(\lambda^a[\lambda^b, \lambda^c]) = 4\mathrm{i} f^{bca}, \qquad \mathrm{Tr}(\lambda^a\{\lambda^b, \lambda^c\}) = 4d^{bca}. \tag{2}$$

From these relations, deduce the complete antisymmetry of $f^{abc}$ and the complete symmetry of $d^{abc}$ with respect to the interchange of two of the indices $a, b, c$.

d) An arbitrary hermitian $3 \times 3$ matrix $K$ can be expressed as $K = k_0\mathbb{1} + \sum_{a=1}^{8} k_a\lambda^a$. “Trace out” the coefficients $k_i$, $i = 0, \dots, 8$, and use the result to derive the relation

$$\sum_{a=1}^{8}(\lambda^a)^i_j(\lambda^a)^k_l = 2\Big(\delta^i_l\delta^k_j - \frac{1}{3}\,\delta^i_j\delta^k_l\Big). \tag{3}$$

*Please turn over!*

**Exercise 8.3** *SU(3) colour symmetry and colour confinement in quantum chromodynamics* (3 points)

The theory of strong interactions, *quantum chromodynamics*, attributes *colour charges* to the fundamental constituents (*quarks* and *antiquarks*) of strongly interacting matter, where the colour part of a quark state $|q\rangle$ transforms in the fundamental representation 3 of SU(3) and that of an anti-quark state $|\bar{q}\rangle$ transforms in the anti-fundamental representation $3^*$. The most simple approximation of a bound state of quarks and antiquarks is described by SU(3)-invariant tensor product states, e.g. of the form

$$|M_{q\bar{q}}\rangle = \sum_{i,j=1}^{3} c_i^j |q^i\rangle \otimes |\bar{q}_j\rangle, \tag{4}$$

$$|B_{qqq}\rangle = \sum_{i,j,k=1}^{3} c_{ijk} |q^i\rangle \otimes |q^j\rangle \otimes |q^k\rangle, \tag{5}$$

where $c_i^j$ and $c_{ijk}$ are some appropriate complex numbers, $|q^i\rangle$ denote the (orthonormalised) basis states of 3, and $|\bar{q}_j\rangle$ the basis states of $3^*$. The necessity of this colour SU(3) invariance is known as the principle of *colour confinement*.

a) Mesons are $q\bar{q}$ bound states. Determine the corresponding constants $c_{ij}$ in (4), fixing an overall normalisation constant according to $\|M_{q\bar{q}}\| = 1$.

b) Baryons are $qqq$ bound states. Determine the corresponding constants $c_{ijk}$ in (5), fixing an overall normalisation constant according to $\|B_{qqq}\| = 1$.

c) Which types of the following states can exist as bound states according to colour confinement: $qq$, $qq\bar{q}\bar{q}$, $qqqq\bar{q}$?