Exercises to Relativistic Quantum Field Theory — Sheet 4

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Exercise 4.1 *Non-relativistic propagator reloaded* (2 points)

The retarded and advanced propagators $G_{\mathrm{ret/adv}}$ for a Schrödinger wave function $\psi(x) = \psi(t,\vec{x})$ with $x = (t \equiv x_0, \vec{x})$ are defined by

$$\theta(x_0 - y_0)\,\psi(x) = +i \int \mathrm{d}^3y\, G_{\mathrm{ret}}(x-y)\,\psi(y),$$
$$\theta(y_0 - x_0)\,\psi(x) = -i \int \mathrm{d}^3y\, G_{\mathrm{adv}}(x-y)\,\psi(y).$$

a) Derive the condition

$$\left(i\partial_t + \frac{1}{2m}\triangle - V(x)\right) G_{\mathrm{ret/adv}}(x) = \delta(x)$$

from Schrödinger's equation for $\psi(x)$.

b) In Exercise 3.1 you have calculated the Fourier-transformed propagators

$$\tilde{G}_{0,\mathrm{ret/adv}}(p) = \frac{1}{p_0 - \frac{\vec{p}^2}{2m}}$$

with $p = (p_0, \vec{p})$ of the free theory (i.e. for $V(x) = 0$) from the ansatz

$$G_{\mathrm{ret/adv}}(x) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\, \mathrm{e}^{-ipx}\, \tilde{G}_{\mathrm{ret/adv}}(p). \tag{1}$$

Now perform the p_0-integration in Eq. (1), after shifting the pole in p_0 according to $\tilde{G}_{0,\mathrm{ret/adv}}(p) = (p_0 - \vec{p}^2/(2m) \pm i\epsilon)^{-1}$ with an infinitesimal $\epsilon > 0$. Which signs of $\pm i\epsilon$ correspond to the retarded and advanced cases?
(*Hint:* show that and use $\theta(\pm\tau) = \frac{\mp 1}{2\pi i}\int_{-\infty}^{+\infty} \mathrm{d}\omega\, \frac{\mathrm{e}^{-i\omega\tau}}{\omega \pm i\epsilon}$.)

c) Calculate $G_{0,\mathrm{ret/adv}}(x)$ explicitly upon carrying out the integration over d^3p in (1).
(*Hint:* use the auxiliary integral $\int_{-\infty}^{+\infty} \mathrm{d}z\, \mathrm{e}^{-a(z+b)^2} = \sqrt{\pi/a}$, where $b \in \mathbb{R}$, $a \in \mathbb{C}$, $a \neq 0$, $\mathrm{Re}(a) \geq 0$.)

Please turn over!

Exercise 4.2 *Electromagnetic interaction of charged scalars* (2 points)

Generically the interaction of the electromagnetic field $A^\mu = (\phi, \vec{A})$ with charged fields is described by a Lagrangian of the form

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - j^\mu A_\mu + \mathcal{L}_0,$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the electromagnetic field-strength tensor, $j^\mu = (\rho, \vec{j})$ the 4-current density of the charges, and $\mathcal{L}_0$ the Lagrangian for the free propagation of the charges (and if relevant of other interactions among them), i.e. $\mathcal{L}_0$ does not depend on A^μ.

(*Comment:* in relativistic field theory it is customary to use Lorentz-Heaviside units, which result from the SI units upon setting $\mu_0 = \varepsilon_0 = c = 1$.)

a) Verify the homogeneous Maxwell equations for the electric field $\vec{E} = -\nabla\phi - \dot{\vec{A}}$ and the magnetic flux density $\vec{B} = \nabla \times \vec{A}$ from the definition of $F^{\mu\nu}$.

b) Derive the inhomogeneous Maxwell equations for the field strength in their covariant form $\partial_\mu F^{\mu\nu} = j^\nu$ from the Euler-Lagrange equations for A^μ and bring them into their usual form in terms of $\vec{E}$ and $\vec{B}$. Verify current conservation $\partial_\mu j^\mu = 0$.

c) Now consider a complex scalar field Φ to describe a spinless particle with electric charge q and mass m, as in Exercise 3.3b). The free propagation of Φ is described by

$$\mathcal{L}_0(\Phi, \partial\Phi) = (\partial\Phi)^*(\partial\Phi) - m^2 \Phi^* \Phi.$$

The electromagnetic interaction between Φ and A^μ is introduced by the "minimal substitution" $\partial^\mu \to D^\mu = \partial^\mu + iqA^\mu$ in $\mathcal{L}_0$, resulting in

$$\mathcal{L}_\Phi(\Phi, \partial\Phi, A) = \mathcal{L}_0(\Phi, D\Phi) = \mathcal{L}_0(\Phi, \partial\Phi) - j_\mu A^\mu.$$

Derive the explicit form of the current density j^μ.

d) $\mathcal{L}_\Phi$ is invariant under the global transformation $\Phi \to \Phi' = \exp(-iq\omega)\Phi$, with ω denoting an arbitrary real number. Derive the Noether current corresponding to this symmetry and compare it with j^μ from above.