Exercises to Relativistic Quantum Field Theory — Sheet 7

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Exercise 7.1 *Wick theorem for bosonic fields* (2 points)

The purpose of this exercise is to prove Wick's theorem for bosonic, real, free field operators $\phi_i \equiv \phi_i(x_i)$, which states that

$$T[\phi_1 \cdots \phi_n] = {:}\phi_1 \cdots \phi_n{:} + \sum_{\text{pairs } ij} {:}\phi_1 \cdots \wick{\c{\phi}_i \cdots \c{\phi}_j} \cdots \phi_n{:}$$
$$+ \sum_{\text{double pairs } ij,kl} {:}\phi_1 \cdots \wick{\c1{\phi}_i \cdots \c2{\phi}_k \cdots \c1{\phi}_j \cdots \c2{\phi}_l} \cdots \phi_n{:} + \ldots$$

with the contractions representing propagators, $\wick{\c{\phi}_i \c{\phi}_j} = \langle 0|T[\phi_i \phi_j]|0\rangle$. For $n = 2$ the theorem has already been proven in Exercise 6.2. We organise the general proof in two steps. Without loss of generality we can assume that $t_n = x_n^0$ is the smallest time variable, i.e. $T[\phi_1 \cdots \phi_n] = T[\phi_1 \cdots \phi_{n-1}]\phi_n$.

a) First prove the lemma

$$ {:}\phi_1 \cdots \phi_{n-1}{:}\, \phi_n = {:}\phi_1 \cdots \phi_n{:} + \sum_{k=1}^{n-1} {:}\phi_1 \cdots \wick{\c{\phi}_k \cdots \c{\phi}_n}{:}\,. $$

To this end, split ϕ_n according to $\phi_n = \phi_n^{(+)} + \phi_n^{(-)}$ into their positive and negative frequency parts $\phi_n^{(\pm)}$, i.e. $\phi_n^{(+)}$ involves only annihilation operators and $\phi_n^{(-)}$ only creation operators. For $\phi_n^{(+)}$ the lemma is trivially verified, for $\phi_n^{(-)}$ you can proceed via induction in n.

b) Argue that the lemma of a) trivially generalises to cases where contractions already appear inside the normal orderings, for example:

$$ {:}\phi_1 \cdots \wick{\c{\phi}_i \cdots \c{\phi}_j} \cdots \phi_{n-1}{:}\, \phi_n = {:}\phi_1 \cdots \wick{\c{\phi}_i \cdots \c{\phi}_j} \cdots \phi_n{:} $$
$$ + \sum_{k=1}^{n-1} {:}\phi_1 \cdots \wick{\c1{\phi}_i \cdots \c2{\phi}_k \cdots \c1{\phi}_j \cdots \c2{\phi}_n}{:}\,. $$

c) Prove Wick's theorem via induction in n using the result of b).

Please turn over!

Exercise 7.2 *S-operator for two interacting scalar fields* (1 point)

Consider a theory of a complex scalar field ϕ (particle ϕ and antiparticle $\bar{\phi}$) and a real scalar field Φ (particle Φ) with the Lagrangian density given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \frac{1}{2}M^2\Phi^2 + (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2\phi^\dagger\phi + \mathcal{L}_{\text{int}},$$

where $\mathcal{L}_{\text{int}} = \lambda\phi^\dagger\phi\,\Phi$. Expand the S-operator,

$$S = T\exp\left(i\int \mathrm{d}^4x\,\mathcal{L}_{\text{int}}(x)\right),$$

up to order λ^2 and use Wick's theorem to express the result in terms of propagators and normal-ordered products of fields. Note that the λ^n contribution can be written in the form

$$\frac{1}{n!}\int \mathrm{d}^4x_1 \dots d^4x_n : \dots :$$

Represent the result diagrammatically using the following notation:

- External lines:

$\phi^\dagger(x) =$ , $\phi(x) =$ , $\Phi(x) =$ 

- Internal lines:

$\overline{\phi(x_1)\phi^\dagger(x_2)} =$ , $\overline{\Phi(x_1)\Phi(x_2)} =$ 

- Vertices:

$i\lambda =$