Exercises to Relativistic Quantum Field Theory — Sheet 8
Prof. S. Dittmaier, Universität Freiburg, WS 2019/20

Exercise 8.1 *S-operator for two interacting scalar fields (cont'd)* (1 point)

Consider again the field theory of a complex scalar field ϕ (particle ϕ and antiparticle $\bar{\phi}$) and a real scalar field Φ (particle Φ) from Exercise 7.2 with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \frac{1}{2} M^2 \Phi^2 + (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2 \phi^\dagger \phi + \mathcal{L}_{\text{int}},$$

where $\mathcal{L}_{\text{int}} = \lambda \phi^\dagger \phi\, \Phi$, and make use of the perturbative expansion worked out there.

a) Calculate the S-matrix element $S_{fi} = \langle f|S|i\rangle$ in lowest non-vanishing order between the initial state $|i\rangle = a_\Phi^\dagger(k)|0\rangle$ and the final state $|f\rangle = a_\phi^\dagger(p_1)\, b_\phi^\dagger(p_2)|0\rangle$, where $a_\Phi^\dagger(q)$, $a_\phi^\dagger(q)$, $b_\phi^\dagger(q)$ are the creation operators of the particles Φ, ϕ, and $\bar{\phi}$, respectively.

b) Assuming $M > 2m$, calculate the lowest-order decay width

$$\Gamma_{\Phi\to\phi\bar{\phi}} = \frac{1}{2M} \int \mathrm{d}\Phi_2\, |\mathcal{M}_{fi}|^2$$

for the decay $\Phi \to \phi\bar{\phi}$, where Φ_2 is the 2-particle phase space of the final state (see Exercise 5.2) and the transition matrix element $\mathcal{M}_{fi}$ is related to S_{fi} by

$$S_{fi} = (2\pi)^4 \delta^{(4)}(k - p_1 - p_2)\, \mathrm{i}\mathcal{M}_{fi}.$$

Exercise 8.2 *Fundamental representations of the Lorentz group* (2 points)

The general form of Lorentz transformations in the two fundamental representations of the Lorentz group is given by

$$\Lambda_{\mathrm{R}} = \exp\left(-\tfrac{\mathrm{i}}{2}(\vec{\phi} + \mathrm{i}\vec{\nu}) \cdot \vec{\sigma}\right), \qquad \Lambda_{\mathrm{L}} = \exp\left(-\tfrac{\mathrm{i}}{2}(\vec{\phi} - \mathrm{i}\vec{\nu}) \cdot \vec{\sigma}\right)$$

with the real group parameters $\vec{\phi} = (\phi_1, \phi_2, \phi_3)^{\mathrm{T}}$, $\vec{\nu} = (\nu_1, \nu_2, \nu_3)^{\mathrm{T}}$ and the Pauli matrices $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)^{\mathrm{T}}$.

a) Show that $\Lambda_{\mathrm{R}}^\dagger = \Lambda_{\mathrm{L}}^{-1}$ and $\Lambda_{\mathrm{L}}^\dagger = \Lambda_{\mathrm{R}}^{-1}$.

b) Show that $\det(\Lambda_{\mathrm{R}}) = \det(\Lambda_{\mathrm{L}}) = 1$ using $\det(\exp(A)) = \exp(\mathrm{Tr}(A))$ for a matrix A.

c) Which transformations are characterised by $\Lambda_{\mathrm{R/L}}^\dagger = \Lambda_{\mathrm{R/L}}$, which by $\Lambda_{\mathrm{R/L}}^\dagger = \Lambda_{\mathrm{R/L}}^{-1}$?

d) Calculate Λ_{R} and Λ_{L} for a pure boost in the direction $\vec{e}$, $|\vec{e}| = 1$, i.e. with $\vec{\nu} = \nu\vec{e}$, $\vec{\phi} = 0$, and for a pure rotation around the axis $\vec{e}$, i.e. with $\vec{\phi} = \phi\vec{e}$, $\vec{\nu} = 0$.

Please turn over!

Exercise 8.3 *Connection between Λ_{R}, Λ_{L}, and $\Lambda^{\mu}{}_{\nu}$* (1 point)

The general matrix representing a Lorentz transformation of a four-vector is given by

$$\Lambda^{\mu}{}_{\nu} = \exp\left(-\frac{\mathrm{i}}{2}\omega_{\alpha\beta}M^{\alpha\beta}\right)^{\mu}{}_{\nu} \qquad \text{with} \qquad (M^{\alpha\beta})^{\mu}{}_{\nu} = \mathrm{i}(g^{\alpha\mu}g^{\beta}{}_{\nu} - g^{\beta\mu}g^{\alpha}{}_{\nu})$$

and the antisymmetric parameters $\omega_{jk} = \epsilon_{jkl}\phi_l$ and $\omega_{0j} = -\omega_{j0} = \nu_j$. The connection between Λ_{R}, Λ_{L} (see Exercise 8.2) and $\Lambda^{\mu}{}_{\nu}$ is

$$\Lambda_{\mathrm{R}}^{\dagger}\sigma^{\mu}\Lambda_{\mathrm{R}} = \Lambda^{\mu}{}_{\nu}\sigma^{\nu}, \qquad \Lambda_{\mathrm{L}}^{\dagger}\bar{\sigma}^{\mu}\Lambda_{\mathrm{L}} = \Lambda^{\mu}{}_{\nu}\bar{\sigma}^{\nu},$$

where $\sigma^{\mu} = (\mathbb{1}, \sigma^1, \sigma^2, \sigma^3)$ and $\bar{\sigma}^{\mu} = (\mathbb{1}, -\sigma^1, -\sigma^2, -\sigma^3)$. Verify these relations for infinitesimal transformations with the parameters $\delta\phi_k$, $\delta\nu_k$.