Exercises to Relativistic Quantum Field Theory — Sheet 11

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Exercise 11.1 *Pair production of scalars in the Yukawa model* (2 points)

Consider the pair production of two identical, neutral scalar particles S that are produced via fermion–antifermion annihilation,

$$f(p_1) + \bar{f}(p_2) \;\to\; S(k_1) + S(k_2),$$

where the momentum assignment of the respective particles is indicated in brackets. For simplicity, the fermions are considered massless. In the centre-of-mass system the momenta are given by

$$(p^\mu_{1,2}) = E(1,0,0,\pm 1), \qquad (k^\mu_{1,2}) = E(1, \pm\beta_S \sin\theta\cos\varphi, \pm\beta_S \sin\theta\sin\varphi, \pm\beta_S\cos\theta),$$

where E is the beam energy and $\beta_S = \sqrt{1 - m_S^2/E^2}$ is the velocity of the scalars of mass m_S. The Dirac fermion f (field ψ) and the scalar S (field ϕ) interact via a pure Yukawa interaction described by the Lagrangian

$$\mathcal{L}_I = -y\bar{\psi}\psi\phi,$$

with y denoting a (dimensionless) coupling constant.

a) Draw all relevant Feynman diagrams for the transition matrix element $\mathcal{M}$ in lowest perturbative order and write down the explicit expression for $\mathcal{M}$. How does $\mathcal{M}$ behave under the interchange $k_1 \leftrightarrow k_2$ and why?

b) Calculate the spin-averaged squared transition matrix element $\overline{|\mathcal{M}|^2} = \frac{1}{4}\sum_{\text{pol}} |\mathcal{M}|^2$ and show that

$$\overline{|\mathcal{M}|^2} = \frac{y^4}{2}\left(\frac{1}{t} - \frac{1}{u}\right)^2 \left(ut - m_S^4\right).$$

c) Derive both the differential cross section $\mathrm{d}\sigma/\mathrm{d}\cos\theta$ and the total cross section σ.

d) Draw all Feynman graphs for $\mathcal{M}$ of order y^4, i.e. in 1-loop approximation, which contribute to this process.

Please turn over!

Exercise 11.2 *Free photon field in radiation gauge* (1 point)

The field operator of the free photon field in radiation gauge ($A^0 = 0, \nabla\vec{A} = 0$) is given by

$$A^\mu(x) = \int \mathrm{d}\tilde{k} \sum_{\lambda=\pm} \left(\mathrm{e}^{-\mathrm{i}kx} \varepsilon^\mu_\lambda(k) a_\lambda(\vec{k}) + \mathrm{e}^{+\mathrm{i}kx} \varepsilon^\mu_\lambda(k)^* a^\dagger_\lambda(\vec{k}) \right) \Big|_{k_0=|\vec{k}|}$$

with the creation and annihilation operators $a^\dagger_\lambda(\vec{k})$ and $a_\lambda(\vec{k})$, which are normalised as in the lecture. The polarisation vectors $\varepsilon^\mu_\pm(k)$ are defined as

$$(\varepsilon^\mu_\pm(\hat{k})) = \frac{1}{\sqrt{2}}(0, 1, \pm\mathrm{i}, 0) \qquad \text{for} \quad (\hat{k}^\mu) = \hat{k}_0(1, 0, 0, 1)$$

and analogously for other directions.

a) Verify the polarisation sum

$$\sum_{\lambda=\pm} \varepsilon^m_\lambda(k)\varepsilon^n_\lambda(k)^* = \delta^{mn} - \frac{k^m k^n}{\vec{k}^2} \qquad \text{for} \quad m, n = 1, 2, 3.$$

b) The field variable that is canonical conjugate to A^m is $\Pi^m = F^{m0}$. Calculate the canonical equal-time commutators, i.e. $[A^m(t,\vec{x}), A^n(t,\vec{y})]$, $[A^m(t,\vec{x}), \Pi^n(t,\vec{y})]$, $[\Pi^m(t,\vec{x}), \Pi^n(t,\vec{y})]$. Make use of the "transverse δ-function" when appropriate,

$$\delta^{mn}_{\mathrm{tr}}(\vec{x}) = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \left(\delta^{mn} - \frac{k^m k^n}{\vec{k}^2} \right) \mathrm{e}^{-\mathrm{i}\vec{k}\vec{x}}, \quad m, n = 1, 2, 3.$$

Exercise 11.3 *Massive gauge-boson propagator in covariant gauge* (1 point)

The propagator $D^{\mu\nu}_\xi(x)$ of a massive vector boson of mass M in covariant gauge is defined by

$$\left(g_{\mu\nu}(\Box + M^2) + \left(\frac{1}{\xi} - 1\right)\partial_\mu\partial_\nu \right) D^{\nu\rho}_\xi(x) = \delta^\rho_\mu \delta^{(4)}(x).$$

Calculate the Fourier transform $\tilde{D}^{\mu\nu}_\xi(q)$ of the propagator upon inserting

$$D^{\mu\nu}_\xi(x) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \mathrm{e}^{-\mathrm{i}qx} \tilde{D}^{\mu\nu}_\xi(q).$$

Here it is useful to employ the decomposition of $\tilde{D}^{\mu\nu}_\xi(q)$ into its transverse part $\tilde{D}_{T,\xi}(q)$ and its longitudinal part $\tilde{D}_{L,\xi}(q)$, so that

$$\tilde{D}^{\mu\nu}_\xi(q) = \tilde{D}_{T,\xi}(q) \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) + \tilde{D}_{L,\xi}(q) \frac{q^\mu q^\nu}{q^2}.$$

Determine the limit $\xi \to \infty$ of $\tilde{D}^{\mu\nu}_\xi(q)$.