Exercises to Relativistic Quantum Field Theory — Sheet 12

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Exercise 12.1 *"Massive photon" in the Abelian Higgs model* (1.5 points)

Consider a model with a complex scalar field $\phi(x)$ whose dynamics is governed by the Lagrangian

$$\mathcal{L}_\phi(\phi, \partial\phi) = (\partial_\mu\phi)^*(\partial^\mu\phi) - V(\phi^*\phi),$$

where V represents a general potential for the scalar self-interactions. Moreover, the quanta of ϕ carry the electric charge q, so that the full Lagrangian for ϕ including its electromagnetic interaction reads

$$\mathcal{L} = \mathcal{L}_\phi(\phi, D\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

with the covariant derivative $D_\mu = \partial_\mu + \mathrm{i}qA_\mu$ and the field-strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ of the electromagnetic field $A^\mu(x)$.

a) The potential $V(\phi^*\phi)$ is assumed to have a minimum for $|\phi(x)| = v/\sqrt{2}$, so that this condition characterises the ground state of the system (= field configuration of lowest energy). This suggests the following parametrisation of ϕ:

$$\phi(x) = \frac{1}{\sqrt{2}}\Big(v + h(x)\Big)\exp\left(\frac{\mathrm{i}\chi(x)}{v}\right),$$

where $h(x)$ and $\chi(x)$ are real fields. Express $\mathcal{L}$ in terms of $h(x)$, $\chi(x)$, and $A^\mu(x)$.

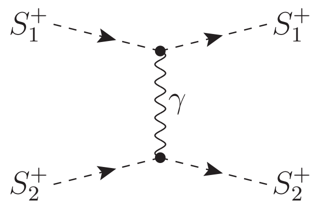
b) Show that the model respects the gauge symmetry $\phi(x) \to \phi'(x) = \mathrm{e}^{-\mathrm{i}q\omega(x)}\phi(x)$, $A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu\omega(x)$ and argue that $\chi(x)$ is an unphysical field, i.e. that it can be consistently set to zero.

c) From $\mathcal{L}$ with $\chi = 0$, read off the part $\mathcal{L}_{AA,0}$ that is responsible for the free motion of A. Derive the Euler-Lagrange equation from $\mathcal{L}_{AA,0}$ for the free motion of A and identify the mass M_A of the quanta of A by comparing the equation of motion with Proca's equation.

Exercise 12.2 *Electromagnetic scattering of two charged scalars* (2 points)

Consider the reaction $S_1^+(p_1) + S_2^+(p_2) \to S_1^+(k_1) + S_2^+(k_2)$ in scalar quantum electrodynamics, i.e. the particles $S_a^\pm$ $(a = 1, 2)$ are scalar particles with electric charges $\mp Q_a e$ and masses M_a. The corresponding Feynman rules are given below. In Born approximation only the following diagram is relevant.

Please turn over!



In the centre-of-mass frame the particle momenta read

$$(p^\mu_{1,2}) = (E_{1,2}, 0, 0, \pm p), \qquad (k^\mu_{1,2}) = (E_{1,2}, \pm p \sin\theta \cos\varphi, \pm p \sin\theta \sin\varphi, \pm p \cos\theta),$$

where $E_{1,2}$ are the energies of the incoming particles, and $p = \sqrt{E_a^2 - M_a^2}$ is the absolute value of their three-momenta.

a) Calculate the squared transition matrix element $|\mathcal{M}|^2$ as a function of the Mandelstam variables $s = (p_1 + p_2)^2$, $t = (p_1 - k_1)^2$, and $u = (p_1 - k_2)^2$.

b) Calculate the differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}|\mathcal{M}|^2.$$

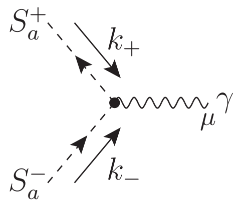
Compare the non-relativistic limit ($M_1, M_2 \gg p$) of the result with the classical Rutherford cross section

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{Rutherford}} = \frac{\alpha^2 Q_1^2 Q_2^2}{4M^2 v^4 \sin^4(\frac{\theta}{2})},$$

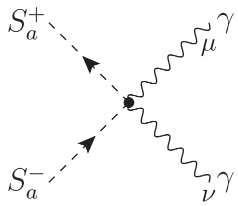
where, in this limit, $v = p/M$ and $M = M_1 M_2/(M_1 + M_2)$ denote the relative velocity and the reduced mass of the two-body system, respectively, and $\alpha = e^2/(4\pi)$ is the fine-structure constant.

Feynman rules for the charged spin-0 particles $S_a^\pm$

- Vertices:

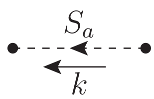


$\mathrm{i}Q_a e(k_+ - k_-)_\mu$

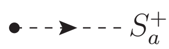


$2\mathrm{i}Q_a^2 e^2 g_{\mu\nu}$

- Propagators and external lines:



$$\frac{\mathrm{i}}{k^2 - M_a^2}$$



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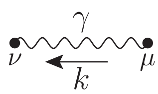

$$\frac{-\mathrm{i}}{k^2}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}(1-\xi)\right)$$

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The fields $S_a^\pm$ are defined to be incoming at the vertices. Outgoing fields $S_a^\pm$ correspond to incoming fields $S_a^\mp$ with reversed momenta.