Introduction to
Relativistic Quantum Field Theory

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Contents

Chapter 1

Introduction

Relativistic quantum field theory
= mathematical framework for description of elementary particles and their interactions

Guiding principles for the construction of field theory and of specific models of interactions:

- relativistic structure of space–time
- principles of quantum mechanics
- empirical knowledge collected in colliders experiments (mainly $\mathrm{e^+e^-}$, $\mathrm{e^\pm p}$, pp, $\mathrm{p\bar{p}}$)

Some empirical facts on particle collisions:

- Particle creation and annihilation is possible in collisions.
- Relativistic kinematics (four-momentum conservation, conversion of mass and energy) is extremely well confirmed.
- The spectrum of observed particles is very rich, but only very few are really elementary:
  - *Leptons* (spin 1/2): e, ν_e, μ, ν_μ, τ, ν_τ
  - *Quarks* (spin 1/2): $\underbrace{\mathrm{u, d, s, c, b,}}_{\textit{confined}\text{ in hadrons, i.e. }\textit{mesons}\ (q\bar{q})\text{ or baryons }(qqq)}$ t
  - *Gauge bosons* (spin 1):
    force carriers of the $\underbrace{\text{strong}}_{\text{gluons (confined)}}$ and $\underbrace{\text{electroweak}}_{\gamma,\ \mathrm{Z^0},\ \mathrm{W^\pm}\text{ bosons}}$ interactions
  - *Higgs boson* (spin 0):
    lends mass to all elementary particles,
    recently discovered (full identification ongoing)

- Four fundamental interactions can be distinguished:
  - *electromagnetic interaction* } *electroweak* interaction
  - *weak interaction*
  - *strong interaction*
  - *gravity* (not accessible by collider experiments)

Features of interaction models

- (special) relativistic covariance
- description of particles by states in Hilbert space
- description of particle dynamics by local fields
- field quantization

} relativistic quantum field theory

- internal symmetries between particles not only global, but also local } gauge theories

The role of symmetries

- space–time symmetry:
  Lorentz/Poincaré covariant formulation of field theory
  → *mass* and *spin* as fundamental properties of particles
- internal symmetries:
  - unification of different particles into *multiplets* of symmetry groups
    → further quantum numbers (charge, isospin, etc.)
  - connection between symmetry and dynamics by *gauging* the symmetry:

$$\text{global symmetry} \xrightarrow[\substack{\text{with own dynamics and} \\ \text{couplings to matter fields}}]{\text{introduction of gauge bosons}} \text{local symmetry}$$

The role of field quantization

- resolution of various inconsistencies in relativistic wave equations
  (negative-energy solutions, probability interpretation, etc.)
- wave-particle dualism
- creation and annihilation of particles
- connection between spin and statistics
  (bosons: spin = 0,1,...; fermions: spin = 1/2, 3/2, ...)

Part I

Quantization of Scalar Fields

Chapter 2

Recapitulation of Special Relativity

2.1 Lorentz transformations, four-vectors, tensors

2.1.1 Minkowski space

Definitions and notation:

- 3-vectors: $\vec{a} = (a^i)$, Latin indices: $i = 1, ..., 3$
  $\hookrightarrow$ span 3-dim. position space

- *Contravariant* 4-vectors: $a^\mu = (a^0, \vec{a})$, Greek indices: $\mu = 0, ..., 3$
  $\hookrightarrow$ span 4-dim. Minkowski space

- Space–time points (*events*): $x^\mu = (x^0, \vec{x}) = (c\,t, \vec{x})$

  Natural units used in the following: $c \to 1$, $\hbar \to 1$

- *Metric tensor*: $(g_{\mu\nu}) = (g^{\mu\nu}) = \mathrm{diag}(+1, -1, -1, -1)$

  > Comment:
  > Equations like $g^{\mu\nu} = g_{\mu\nu}$—although correct for each coefficient—should be avoided, since the two sides correspond to two different geometrical objects.

- *Covariant* 4-vectors: $a_\mu = (a^0, -\vec{a}) = g_{\mu\nu} a^\nu$, $a^\mu = g^{\mu\nu} a_\nu$

  Note: Einstein's convention used, i.e. summation over pairs of equal upper and lower indices

- *Scalar product*:

$$a \cdot b = a^0 b^0 - \vec{a} \cdot \vec{b} = a^\mu b_\mu = a_\mu b^\mu = g_{\mu\nu} a^\mu b^\nu = g^{\mu\nu} a_\mu b_\nu \tag{2.1}$$

- *Length of 4-vectors*: $a^\mu a_\mu = (a^0)^2 - \vec{a}^{\,2} = g_{\mu\nu} a^\mu a^\nu = ...$

  $\hookrightarrow$ space–time distance s^2 of two events "a" and "b":

$$s^2 = (x_a - x_b)^\mu (x_a - x_b)_\mu = (t_a - t_b)^2 - (\vec{x}_a - \vec{x}_b)^2 \tag{2.2}$$

Basic principles of special relativity

- *relativity principle* $\rightarrow$ laws of physics equivalent in all frames of inertia
- *constancy of speed of light* $\rightarrow$ value of c is equal in all frames

$\Rightarrow$ Scalar products (space–time distances, etc.) independent of frame of reference !

Classification of space–time distances: (independent of reference frame!)

$$(x_a - x_b)^2 = \left\{ \begin{array}{c} \text{time-like} \\ \text{light-like} \\ \text{space-like} \end{array} \right\} \quad \text{if } (x_a-x_b)^2 \left\{ \begin{array}{c} >0 \\ =0 \\ <0 \end{array} \right\}, \text{ i.e. } |t_a-t_b| - |\vec{x}_a - \vec{x}_b| \left\{ \begin{array}{c} >0 \\ =0 \\ <0 \end{array} \right. \tag{2.3}$$

- Time-like: Signals with velocity < speed of light c can be sent from x_a^μ to x_b^μ.
- Light-like: If $t_b > t_a$, a light ray can be sent from x_a^μ to x_b^μ.
- Space-like: There is a frame with $t_a = t_b$, i.e. "a" and "b" happen simultaneously.

All events "b" with $(x_a - x_b)^2 = 0$ form the *light-cone* of x_a.

$\Rightarrow$ The light cone of x_a separates events causally connected/disconnected to "a".

2.1.2 Lorentz transformations

= all coordinate transformations of Minkowski space that leave the space–time distances (2.2) invariant

Definitions:

- *Homogenous Lorentz transformations* = all *linear* transformations

  characterized by 4×4 matrix Λ:

  $$a'^\mu = \Lambda^\mu{}_\nu \, a^\nu, \qquad \text{matrix notation (contravariant vectors!):} \quad a' = \Lambda a \tag{2.4}$$

  obeying the invariance property $a'^2 = a^2$ for all 4-vectors a^μ

  $\Rightarrow$ All scalar products are invariant:

  $$a' \cdot b' = \tfrac{1}{2}(a'+b')^2 - \tfrac{1}{2}a'^2 - \tfrac{1}{2}b'^2 = \tfrac{1}{2}(a+b)^2 - \tfrac{1}{2}a^2 - \tfrac{1}{2}b^2 = a \cdot b \tag{2.5}$$

  $\Rightarrow$ Invariance property of Λ:

  $$g_{\mu\nu} a'^\mu b'^\nu = g_{\mu\nu} \Lambda^\mu{}_\rho \, \Lambda^\nu{}_\sigma \, a^\rho b^\sigma \stackrel{!}{=} g_{\rho\sigma} a^\rho b^\sigma \qquad \Rightarrow \qquad g_{\mu\nu} \Lambda^\mu{}_\rho \, \Lambda^\nu{}_\sigma \stackrel{!}{=} g_{\rho\sigma}, \quad \Lambda^T g \Lambda = g \tag{2.6}$$

- *Inhomogenous Lorentz transformations (Poincaré transformations)*

  = all *affine* transformations of space–time

  characterized by 4×4 matrix Λ and 4-vector a:

  $$x'^\mu = \Lambda^\mu{}_\nu \, x^\nu + a^\mu, \qquad x' = \Lambda x + a \tag{2.7}$$

  $\Rightarrow$ At least all space–time distances invariant: $(x'_a - x'_b)^2 = (x_a - x_b)^2$

Examples:

- Rotations:

$$\Lambda_D = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \qquad \text{with} \quad D^T D = 1, \quad \text{so that} \quad \Lambda_D^T g \Lambda_D = g \tag{2.8}$$

  Rotation around the x^3 axis:

$$D_3(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.9}$$

- *Boosts* (relating inertial frames moving with a relative velocity $\vec{v}$):

  Boost B_3 in the x^3 direction:

$$\begin{aligned} t' &= \gamma(t + v\,x^3), \qquad \gamma = 1/\sqrt{1-v^2} \\ x'^1 &= x^1, \\ x'^2 &= x^2, \\ x'^3 &= \gamma(x^3 + v\,t) \end{aligned} \tag{2.10}$$

  Convenient parametrization of Λ_{B_3} by *rapidity* ν, where $v = \tanh\nu$:

$$\Lambda_{B_3} = \begin{pmatrix} \gamma & 0 & 0 & \gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma v & 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} \cosh\nu & 0 & 0 & \sinh\nu \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\nu & 0 & 0 & \cosh\nu \end{pmatrix} = \Lambda_{B_3}(\nu) \tag{2.11}$$

> Comment:
> The angles $\varphi\vec{e}$ ($0 \le \varphi \le \pi$, $|\vec{e}| = 1$) that parametrize all rotations define a compact set (the angles $\pm\pi\vec{e}$ correspond to the same rotation).
> The rapidities $\nu\vec{e}$ are contained in a non-compact set of numbers ($-\infty < \nu < \infty$).
> $\Rightarrow$ The Lorentz transformations form a *non-compact Lie group*. See also below.

Inverse Lorentz transformations

$a'^\mu = \Lambda^\mu{}_\nu\, a^\nu$, i.e. $a^\mu = (\Lambda^{-1})^\mu{}_\nu\, a'^\nu$

Proposition:

$$(\Lambda^{-1})^\mu{}_\nu = g_{\nu\alpha}\,\Lambda^\alpha{}_\beta\, g^{\beta\mu} \equiv \Lambda_\nu{}^\mu \tag{2.12}$$

Proof:
Verify $\Lambda^{-1}\Lambda = \mathbf{1}$ ($\Lambda\Lambda^{-1} = \mathbf{1}$ analogously):

$$\Lambda_\nu{}^\mu \Lambda^\nu{}_\rho = g_{\nu\alpha}\,\Lambda^\alpha{}_\beta\, g^{\beta\mu}\,\Lambda^\nu{}_\rho = (g_{\nu\alpha}\,\Lambda^\alpha{}_\beta\,\Lambda^\nu{}_\rho)\, g^{\beta\mu} \overset{(2.6)}{=} g_{\beta\rho}\, g^{\beta\mu} = g^\mu_\rho = \delta^\mu_\rho \tag{2.13}$$

q.e.d.

Application: Lorentz transformation of covariant 4-vectors

$$x'_\mu = g_{\mu\nu} x'^\nu = g_{\mu\nu}\Lambda^\nu{}_\rho\, x^\rho = g_{\mu\nu}\,\Lambda^\nu{}_\rho\, g^{\rho\sigma} x_\sigma = \Lambda_\mu{}^\sigma x_\sigma. \tag{2.14}$$

2.1.3 Differential operators

Definitions:

- *covariant 4-gradiant*:

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right) \tag{2.15}$$

- *contravariant 4-gradiant*:

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = g^{\mu\nu}\partial_\nu = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right) \tag{2.16}$$

- *wave (d'Alembert) operator*:

$$\Box \equiv \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \tag{2.17}$$

Lorentz/Poincaré transformation properties: $x'^\mu = \Lambda^\mu{}_\nu \, x^\nu + a^\mu$

- 4-gradients:

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = \Lambda_\mu{}^\nu \, \partial_\nu, \qquad \partial'^\mu = ... = \Lambda^\mu{}_\nu \, \partial^\nu \tag{2.18}$$

- wave operator:

$$\Box' = \partial'_\mu \partial'^\mu = \Lambda_\mu{}^\nu \, \Lambda^\mu{}_\rho \, \partial_\nu \partial^\rho = g^\nu_\rho \, \partial_\nu \partial^\rho = \partial_\nu \partial^\nu = \Box = \text{invariant} \tag{2.19}$$

- *4-divergence* of a vector field V^μ:

$$\partial_\mu V^\mu(x) = \partial_0 V^0(x) + \partial_i V^i(x) = \dot{V}^0(x) + \vec{\nabla} \cdot \vec{V} = \text{invariant} \tag{2.20}$$

2.1.4 Tensors

Definitions

- *Contravariant tensor* $T^{\mu_1 \cdots \mu_n}$ of rank n = object that transforms like the direct product $a^{\mu_1} \ldots a^{\mu_n}$ under the change of coordinate frames, i.e.

$$T'^{\mu_1 \cdots \mu_n} = \Lambda^{\mu_1}{}_{\rho_1} \ldots \Lambda^{\mu_n}{}_{\rho_n} T^{\rho_1 \cdots \rho_n}. \tag{2.21}$$

- A *covariant tensor* $T_{\mu_1 \cdots \mu_n}$ of rank n transforms like $a_{\mu_1} \ldots a_{\mu_n}$.

- A *mixed-rank* (n, m) *tensor* transforms as

$$T'^{\mu_1 \cdots \mu_n}{}_{\nu_1 \ldots \nu_m} = \Lambda^{\mu_1}{}_{\rho_1} \ldots \Lambda^{\mu_n}{}_{\rho_n} \Lambda_{\nu_1}{}^{\sigma_1} \ldots \Lambda_{\nu_m}{}^{\sigma_m} T^{\rho_1 \cdots \rho_n}{}_{\sigma_1 \ldots \sigma_m}. \tag{2.22}$$

Invariant tensors:

- Metric tensor: $g'^{\mu\nu} = g^{\mu\nu}, \quad g'_{\mu\nu} = g_{\mu\nu}$

- Totally antisymmetric tensor:

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } (\mu\nu\rho\sigma) = \text{even permutation of } (0123) \\ -1 & \text{if } (\mu\nu\rho\sigma) = \text{odd permutation of } (0123) \\ \;\;0 & \text{otherwise} \end{cases} \tag{2.23}$$

  Transformation:

$$\epsilon'^{\mu\nu\rho\sigma} = \epsilon^{\mu\nu\rho\sigma} \times \det \Lambda = \pm \epsilon^{\mu\nu\rho\sigma} = \text{invariant pseudo-tensor} \tag{2.24}$$

(see Exercise 1.1)

2.2 Lorentz group and Lorentz algebra

2.2.1 Classification of Lorentz transformations

Definition: The set of Lorentz transformations forms the *Lorentz group* L:

- closure: $\Lambda_1 \Lambda_2 = \Lambda =$ Lorentz transformation (prove!)

- associativity: $\Lambda_1(\Lambda_2\Lambda_3) = (\Lambda_1\Lambda_2)\Lambda_3$

- unit element: $(\Lambda_e)^\mu{}_\nu = \delta^\mu_\nu$

- inverse elements: $(\Lambda^{-1})^\mu{}_\nu = \Lambda_\nu{}^\mu$

Important discrete Lorentz tansformations:

- *parity* P (space inversion):

$$x^\mu \to x'^\mu = \Lambda_P{}^\mu{}_\nu x^\nu = \begin{pmatrix} x^0 \\ -\vec{x} \end{pmatrix}, \qquad \Lambda_P = \mathrm{diag}(+1,-1,-1,-1) \tag{2.25}$$

- time reversal T:

$$x^\mu \to x'^\mu = \Lambda_T{}^\mu{}_\nu x^\nu = \begin{pmatrix} -x^0 \\ \vec{x} \end{pmatrix}, \qquad \Lambda_T = \mathrm{diag}(-1,+1,+1,+1) \tag{2.26}$$

Invariant properties of Λ matrices and classification:

- $\det\Lambda = \pm 1$, since $\Lambda^T g \Lambda = g$ $\quad\Rightarrow$ Def.: $\quad L_\pm \equiv \{\Lambda |\det\Lambda = \pm 1\}$
  L_+ = subgroup of *proper* Lorentz transformations $\quad$ ($L_- \neq$ subgroup)

- $|\Lambda^0{}_0| \geq 1$, since $g_{00} = 1 = g_{\mu\nu}\Lambda^\mu{}_0\,\Lambda^\nu{}_0 = (\Lambda^0{}_0)^2 - (\Lambda^i{}_0)^2$
  Def.: $\quad L^\uparrow \equiv \{\Lambda | \Lambda^0{}_0 \geq 1\}, \qquad L^\downarrow \equiv \{\Lambda | \Lambda^0{}_0 \leq -1\}$
  $L^\uparrow$ = subgroup of *orthochronous* Lorentz transformations $\quad$ ($L^\downarrow \neq$ subgroup)

- Consequence: break-up of the Lorentz group into four disconnected subsets

$$\begin{array}{llll} & \det\Lambda: & \Lambda^0{}_0: & \text{Example:} \\ L_+^\uparrow & + & >1 & \Lambda = \mathbf{1} \\ L_-^\uparrow & -1 & >1 & \Lambda = \Lambda_P \\ L_-^\downarrow & -1 & <-1 & \Lambda = \Lambda_T \\ L_+^\downarrow & +1 & <-1 & \Lambda = -\mathbf{1} = \Lambda_P\Lambda_T \end{array} \tag{2.27}$$

  Def.: $\quad L_+^\uparrow \equiv \{\Lambda |\det\Lambda = +1, \Lambda^0{}_0 \geq 1\}$
  $=$ group of *proper, orthochronous (special)* Lorentz transformations

- Decomposition of Λ (non-trivial!):
  Each $\Lambda \in L_+^\uparrow$ can be written as a product of a rotation and a boost:

$$\Lambda = \Lambda_B \Lambda_D. \tag{2.28}$$

  The rotations form a subgroup of $L_+^\uparrow$, while the boosts do not.

2.2.2 Infinitesimal transformations and group generators

Infinitesimal rotations and boosts:

Consider the rotation (2.9) and the boost (2.11) for infinitesimal parameters $\delta\varphi$ and $\delta\nu$:

$$\Lambda_{D_3}(\delta\varphi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\delta\varphi & 0 \\ 0 & \delta\varphi & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \mathcal{O}(\delta\varphi^2) \equiv \mathbf{1} - \mathrm{i}\delta\varphi\, J^3 + \mathcal{O}(\delta\varphi^2), \tag{2.29}$$

$$\Lambda_{B_3}(\delta\nu) = \begin{pmatrix} 1 & 0 & 0 & \delta\nu \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \delta\nu & 0 & 0 & 1 \end{pmatrix} + \mathcal{O}(\delta\nu^2) \equiv \mathbf{1} - \mathrm{i}\delta\nu\, K^3 + \mathcal{O}(\delta\nu^2), \tag{2.30}$$

General infinitesimal rotations or boosts parametrized by six parameters $\delta\varphi_i$ and $\delta\nu_i$:

$$\Lambda_D(\delta\varphi_i) = \mathbf{1} - \mathrm{i}\delta\vec{\varphi}\cdot\vec{J} = \mathbf{1} - \mathrm{i}\delta\varphi_i J^i, \qquad \Lambda_B(\delta\nu_i) = \mathbf{1} - \mathrm{i}\delta\vec{\nu}\cdot\vec{K} = \mathbf{1} - \mathrm{i}\delta\nu_i\, K^i \tag{2.31}$$

Definitions:

J^i = *generator* of infinitesimal rotations around the x^i axis (*angular momentum*)

K^i = *generator* of infinitesimal boosts in the x^i direction

Properties of the generators:

- Explicitly:

$$J^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix}, \quad J^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.32}$$

$$K^1 = \begin{pmatrix} 0 & \mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K^2 = \begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K^3 = \begin{pmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{pmatrix} \tag{2.33}$$

- Hermiticity:

$$J^{i\dagger} = J^i = \text{hermitian}, \qquad K^{i\dagger} = -K^i = \text{anti-hermitian} \tag{2.34}$$

- Commutation relations:

$$[J^i, J^j] = \mathrm{i}\epsilon^{ijk} J^k \qquad \text{(relations of angular momentum)} \tag{2.35}$$

$$[J^i, K^j] = \mathrm{i}\epsilon^{ijk} K^k \qquad (\vec{K} \text{ transforms as 3-vector operator}) \tag{2.36}$$

$$[K^i, K^j] = -\mathrm{i}\epsilon^{ijk} J^k. \tag{2.37}$$

> Comment:
> The third equation expresses the fact that boosts do not form a subgroup of L, but that the product of two boosts in general involves a rotation (the so-called *Wigner rotation*).

General infinitesimal Lorentz transformations:

- General form: $\Lambda^\mu{}_\nu(\delta\omega) = \delta^\mu_\nu + \delta\omega^\mu{}_\nu + \dots$

  $\hookrightarrow$ condition for Lorentz transformation:

$$\begin{aligned} g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma &= g_{\mu\nu}(\delta^\mu_\rho + \delta\omega^\mu{}_\rho)(\delta^\nu_\sigma + \delta\omega^\nu{}_\sigma) + \dots \\ &= g_{\rho\sigma} + \delta\omega_{\sigma\rho} + \delta\omega_{\rho\sigma} + \dots \stackrel{!}{=} g_{\rho\sigma}, \end{aligned} \tag{2.38}$$

  $\Rightarrow \delta\omega$ are antisymmetric,

$$\delta\omega_{\sigma\rho} = -\delta\omega_{\rho\sigma}, \tag{2.39}$$

  and comprise six independent entries corresponding to $\delta\varphi_i$ and $\delta\nu_i$.

- Generators $M^{\alpha\beta}$:

$$\Lambda^\mu{}_\nu(\delta\omega) = \delta^\mu_\nu + \delta\omega_{\alpha\beta}\, g^{\alpha\mu}\, \delta^\beta_\nu \equiv \delta^\mu_\nu - \frac{\mathrm{i}}{2}\delta\omega_{\alpha\beta}(M^{\alpha\beta})^\mu{}_\nu \tag{2.40}$$

$$\Rightarrow \quad (M^{\alpha\beta})^\mu{}_\nu = \mathrm{i}(g^{\alpha\mu}\delta^\beta_\nu - g^{\beta\mu}\delta^\alpha_\nu) \tag{2.41}$$

  Matrix notation: $\Lambda(\delta\omega) = \mathbf{1} - \frac{\mathrm{i}}{2}\delta\omega_{\alpha\beta}M^{\alpha\beta}$

- Connection between J^i, K^i and $M^{\alpha\beta}$:

$$K^j = M^{0j}\,, \qquad (K^j)^\mu{}_\nu = \mathrm{i}(g^{0\mu}\delta^j_\nu - g^{j\mu}\delta^0_\nu) = \begin{cases} \mathrm{i}\,, & (\mu,\nu) = (0,j) \text{ or } (j,0), \\ 0 & \text{otherwise,} \end{cases} \tag{2.42}$$

$$J^k = \frac{1}{2}\epsilon^{ijk}M^{ij}\,, \quad (J^k)^{mn} = \frac{\mathrm{i}}{2}\epsilon^{ijk}(g^{im}\delta_{jn} - g^{jm}\delta_{in}) = -\mathrm{i}\epsilon^{mnk} \tag{2.43}$$

  (e.g. $J^3 = M^{12}$)

- Commutators: *(Lorentz algebra)*

$$[M^{\mu\nu}, M^{\rho\sigma}] = -\mathrm{i}\,(g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho}) \tag{2.44}$$

> Comment:
> Proof straightforward, easiest based on commutators for J^i, K^i.

Finite Lorentz transformations:

$\hookrightarrow$ result from infinitesimal transformations upon (matrix) exponentiation:

$$\Lambda_D(\vec{\varphi}) = \exp\left(-\mathrm{i}\varphi_i J^i\right), \qquad \Lambda_B(\vec{\nu}) = \exp\left(-\mathrm{i}\nu_i K^i\right), \qquad \Lambda(\omega) \;= \exp\left(-\frac{\mathrm{i}}{2}\omega_{\alpha\beta}M^{\alpha\beta}\right) \tag{2.45}$$

> Comment:
> The iterative limit $\Lambda_D(\varphi_i) = \lim\limits_{N\to\infty}\left(\mathbf{1} - \mathrm{i}\dfrac{\varphi_i}{N}J^i\right)^N$ is demonstrative, but not of much practical use.

2.3 Poincaré group and Poincaré algebra

2.3.1 The basics

Definition: The *Poincaré group* P is the group of all inhomogeneous Lorentz transformations (Λ, a) with $x' = \Lambda x + a$, where $\Lambda \in L$, $a =$ any 4-vector. Obvious restrictions are $P_+^\uparrow$ with $\Lambda \in L_+^\uparrow$, etc.

$\Rightarrow$ P, $P_+^\uparrow$, etc. are non-compact Lie groups with 10 independent parameters.

Subgroups:

- $(\Lambda, 0)$: groups L, $L_+^\uparrow$, etc.
- $(\mathbf{1}, a)$: (abelian) group T_4 of 4-dim. translations

Composition law:

$$(\Lambda_2, a_2)\,(\Lambda_1, a_1) \;=\; (\underbrace{\Lambda_2\Lambda_1}_{\text{as in } L}, \Lambda_2 a_1 + a_2) \tag{2.46}$$

> Comment:
> Proof: Insert $x' = \Lambda_1 x + a_1$ into $x'' = \Lambda_2 x' + a_2$ to obtain $x'' = \Lambda_2(\Lambda_1 x + a_1) + a_2 = \Lambda_2\Lambda_1 x + \Lambda_2 a_1 + a_2$.

$\Rightarrow$ P is *semi-direct* product: $P \;=\; T_4 \rtimes L$

Infinitesimal transformations and generators:

- General transformation:

$$(\Lambda, a) \;=\; \exp\left(-\frac{\mathrm{i}}{2}\omega_{\alpha\beta}M^{\alpha\beta} + \mathrm{i}a_\mu \underbrace{P^\mu}_{\text{generators for translations (\textit{4-momentum})}}\right) \tag{2.47}$$

- Infinitesimal transformation:

$$(\mathbf{1} + \delta\omega, \delta a) \;=\; \mathbf{1} - \frac{\mathrm{i}}{2}\delta\omega_{\alpha\beta}M^{\alpha\beta} + \mathrm{i}\delta a_\mu P^\mu + \ldots \tag{2.48}$$

- Poincaré algebra:

$$[M^{\mu\nu}, M^{\rho\sigma}] \;=\; ..., \quad \text{as in } L_+^\uparrow, \tag{2.49}$$

$$[P^\mu, P^\nu] \;=\; 0, \quad \text{since } T_4 \text{ abelian}, \tag{2.50}$$

$$[P^\mu, M^{\rho\sigma}] \;=\; \mathrm{i}\,(g^{\mu\rho}P^\sigma - g^{\mu\sigma}P^\rho) \tag{2.51}$$

Proof of (2.51):

$$
\begin{aligned}
(\mathbf{1}+\delta\omega,0)^{-1}\,(\mathbf{1},\delta a)\,(\mathbf{1}+\delta\omega,0) &= \left(\mathbf{1}+\frac{\mathrm{i}}{2}\delta\omega_{\alpha\beta}M^{\alpha\beta}\right)(\mathbf{1}+\mathrm{i}\delta a_\mu P^\mu)\left(\mathbf{1}-\frac{\mathrm{i}}{2}\delta\omega_{\alpha\beta}M^{\alpha\beta}\right)+\dots \\
&= \mathbf{1}+\mathrm{i}\delta a_\mu P^\mu-\frac{1}{2}\delta\omega_{\alpha\beta}\delta a_\mu[M^{\alpha\beta},P^\mu]+\dots \\
(\mathbf{1}+\delta\omega,0)^{-1}\,(\mathbf{1},\delta a)\,(\mathbf{1}+\delta\omega,0) &= (\mathbf{1}-\delta\omega,0)\,(\mathbf{1}+\delta\omega,\delta a) \\
&= (\mathbf{1},\delta a^\mu-\delta\omega^{\mu\nu}\delta a_\nu) \\
&= \mathbf{1}+\mathrm{i}(\delta a^\mu-\delta\omega^{\mu\nu}\delta a_\nu)P_\mu+\dots \\
&= \mathbf{1}+\mathrm{i}\delta a_\mu P^\mu-\frac{\mathrm{i}}{2}\delta\omega_{\mu\nu}(\delta a^\nu P^\mu-\delta a^\mu P^\nu)+\dots
\end{aligned}
$$

$\hookrightarrow$ (2.51) follows upon comparing coefficients for arbitrary $\delta\omega_{\alpha\beta}\delta a_\mu$.

q.e.d.

2.3.2 Generators as differential operators

Inspect operation of transformations (Λ, a) on some scalar function $\phi(x)$:

$$\phi(x) \underset{(\Lambda,a)}{\longrightarrow} \phi'(x') = \phi'(\Lambda x+a) \overset{!}{=} \phi(x), \qquad \text{i.e.} \quad \phi'(x) = \phi\Big(\Lambda^{-1}(x-a)\Big) \tag{2.52}$$

- Translations:
  - infinitesimal:

$$
\begin{aligned}
\phi'(x) &= \phi(x-\delta a) = \phi(x)-\delta a^\mu\partial_\mu\phi(x)+\dots \\
&= [1+\mathrm{i}\delta a^\mu(\mathrm{i}\partial_\mu)+\cdots]\,\phi(x) \\
&\equiv [1+\mathrm{i}\delta a^\mu P_\mu+\cdots]\,\phi(x),
\end{aligned} \tag{2.53}
$$

$\Rightarrow$ $P^\mu = \mathrm{i}\partial^\mu = (\mathrm{i}\partial_t, -\mathrm{i}\vec{\nabla})$ = 4-momentum operator as differential operator

  - finite transformations:

$$\phi'(x) = \phi(x-a) = \exp\{\mathrm{i}a_\mu P^\mu\}\phi(x) \tag{2.54}$$

- Homogeneous Lorentz transformations:
  - infinitesimal:

$$
\begin{aligned}
\phi'(x) &= \phi\Big(\Lambda^{-1}x\Big) = \phi\Big(x^\mu+\frac{\mathrm{i}}{2}\delta\omega_{\alpha\beta}(M^{\alpha\beta})^\mu{}_\nu x^\nu+\cdots\Big) \\
&= \phi(x)+\frac{\mathrm{i}}{2}\delta\omega_{\alpha\beta}\underbrace{(M^{\alpha\beta})^\mu{}_\nu}_{=\mathrm{i}(g^{\alpha\mu}\delta^\beta_\nu-g^{\beta\mu}\delta^\alpha_\nu)}x^\nu\partial_\mu\phi(x)+\cdots \\
&= \phi(x)+\frac{\mathrm{i}}{2}\delta\omega_{\alpha\beta}\mathrm{i}(x^\beta\partial^\alpha-x^\alpha\partial^\beta)\phi(x)+\cdots \\
&\equiv \phi(x)-\frac{\mathrm{i}}{2}\delta\omega_{\alpha\beta}L^{\alpha\beta}\phi(x)+\cdots
\end{aligned} \tag{2.55}
$$

$\Rightarrow$ $L^{\alpha\beta} = \mathrm{i}(x^\alpha\partial^\beta-x^\beta\partial^\alpha) = x^\alpha P^\beta-x^\beta P^\alpha$
= generalized angular momentum operator as differential operator

– finite transformations:

$$\phi'(x) \;=\; \phi\Big(\Lambda^{-1}x\Big) \;=\; \exp\Big\{-\frac{\mathrm{i}}{2}\omega_{\alpha\beta}L^{\alpha\beta}\Big\}\phi(x) \tag{2.56}$$

- General case:

$$\phi'(x) \;=\; \phi\Big(\Lambda^{-1}(x-a)\Big) \;=\; \exp\Big\{\mathrm{i}a_\mu P^\mu - \frac{\mathrm{i}}{2}\omega_{\alpha\beta}L^{\alpha\beta}\Big\}\phi(x) \tag{2.57}$$

2.4 Relativistic point particles

Point particle in non-relativistic mechanics: momentum $= m\dot{\vec{x}}$

But: $\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\right) = (1, \dot{\vec{x}}) \neq$ 4-vector, since $\mathrm{d}t \neq$ invariant

$\hookrightarrow$ $m\dot{\vec{x}}$ is not part of a 4-vector !
(m = mass = invariant particle property = constant !)

Correct relativistic generalization with *4-velocity*:

$$\begin{aligned} u^\mu &= \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}, \quad \mathrm{d}\tau = \mathrm{d}t\sqrt{1-v^2}, \quad v = |\dot{\vec{x}}| \\ &\qquad\qquad\quad = \textit{proper time} \text{ of the particle} \\ &\qquad\qquad\quad = \text{time in particle rest frame (“intrinsic clock”)} \\ &= \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, \frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau}\right) = \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, \dot{\vec{x}}\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = (\gamma, \gamma\dot{\vec{x}}), \quad \gamma = 1/\sqrt{1-v^2} \end{aligned} \tag{2.58}$$

Relativistic 4-momentum:

$$p^\mu = mu^\mu = (m\gamma, m\gamma\dot{\vec{x}}) = \text{4-vector} \tag{2.59}$$

$$\hookrightarrow \text{ invariant square:} \quad p^2 = m^2\gamma^2 - m^2\gamma^2 v^2 = m^2$$

$$\begin{aligned} \Rightarrow p^0 &= p_0 = \sqrt{m^2 + \vec{p}^{\,2}} = m\left(1 + \frac{\vec{p}^{\,2}}{2m^2} + \dots\right) \quad \text{for } |\vec{p}| \ll m \\ &= \text{relativistic (total) energy of particle with mass } m \end{aligned}$$

$$\begin{aligned} \hookrightarrow \quad E_0 &= mc^2 = \text{rest energy} \quad (\text{restoring } c \neq 1 \text{ here}) \\ T &= p_0 - m = \text{kinetic energy} \end{aligned} \tag{2.60}$$

Comments:

- $p^\mu = mu^\mu$ can be directly derived from ansatz $\vec{p} = m\dot{\vec{x}} \cdot f(v)$, demanding momentum conservation in collisions and $f(0) = 1$
- $\vec{p}$ follows from invariance of action $S = \int \mathrm{d}t\, L$ for point particle
- p^0 is Hamilton function of free particle = conserved
- 4-momentum conservation directly follows from translational invariance of Lagrange function / action

see exercises !

Comment:
$c = 1, \hbar = 1 \quad \Rightarrow$ All kinematical quantities are measured in the same unit:

$$[E] = [m] = [p] = [x^{-1}] = [t^{-1}]. \tag{2.61}$$

Useful units in high-energy elementary particle physics:

- energy unit:

$$[E] = \text{Giga electron Volt (GeV)}\,, \qquad \text{GeV}/c^2 = 1.8 \times 10^{-24}\text{g}. \tag{2.62}$$

- unit of length:

$$[x] = \text{Fermi} = \text{fm}. \tag{2.63}$$

Relation between units: $\hbar c = 0.197\text{GeV fm}$

Example: particle decay

$$\begin{array}{rcccc} & \pi^- & \to & \mu^- & \bar{\nu}_\mu \\ & \uparrow & & \uparrow & \uparrow \\ \text{masses:} & m_\pi & & m_\mu & \approx 0 \end{array}$$

Rest frame of π^-:

$$p_\pi^\alpha = (m_\pi, \vec{0}), \qquad p_\pi^2 = m_\pi^2, \tag{2.64}$$

$$p_\mu^\alpha = (E_\mu, \vec{p}_\mu), \qquad p_\mu^2 = E_\mu^2 - \vec{p}_\mu^{\,2} = m_\mu^2, \tag{2.65}$$

$$p_\nu^\alpha = (E_\nu, \vec{p}_\nu), \qquad p_\nu^2 = E_\nu^2 - \vec{p}_\nu^{\,2} = 0 \tag{2.66}$$

energy conservation: $$m_\pi = E_\mu + E_\nu \tag{2.67}$$

momentum conservation: $$\vec{0} = \vec{p}_\mu + \vec{p}_\nu \tag{2.68}$$

(2.68) in (2.65) – (2.66):

$$m_\mu^2 = E_\mu^2 - E_\nu^2 = (E_\mu - E_\nu)(E_\mu + E_\nu) \tag{2.69}$$

$$\underset{(2.67)}{=} (E_\mu - E_\nu) m_\pi \tag{2.70}$$

$$\Rightarrow \quad E_\mu = \frac{m_\pi}{2} + \frac{m_\mu^2}{2m_\pi}, \qquad E_\nu = \frac{m_\pi}{2} - \frac{m_\mu^2}{2m_\pi} \tag{2.71}$$

Note: The direction of the decay is not fixed. π^- (= spin 0, no polarization!) decay isotropically in their rest frame.

Chapter 3

The Klein–Gordon equation

3.1 Relativistic wave equation

Non-relativistic quantum mechanics:

- Spinless particle (*scalar*)
  $\hookrightarrow$ state vector $|\psi(t)\rangle \in$ Hilbert space,
  $\psi(t,\vec{x}) = \langle\vec{x}|\psi(t)\rangle =$ (complex) wave function in position representation

- Observables $\rightarrow$ Hermitian operators

  Examples: position $\hat{\vec{x}}$ and momentum $\hat{\vec{p}}$ in position space
  $\hat{\vec{x}} = \vec{x} =$ multiplicative, $\quad \hat{\vec{p}} = \frac{\hbar}{\mathrm{i}}\vec{\nabla}$

- Correspondence principle: $\quad H_{\text{classical}}(x_i, p_i) \rightarrow \hat{H}(\hat{x}_i, \hat{p}_i) \quad$ with $[\hat{x}_i, \hat{p}_j] = \mathrm{i}\hbar\delta_{ij}$

- Time evolution by Schrödinger equation: $\quad \mathrm{i}\hbar\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle$

- Free, spinless particle: $\quad \hat{H} = \frac{\hat{\vec{p}}^{\,2}}{2m}$
  $\hookrightarrow$ Schrödinger equation is wave equation in position space:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(t,\vec{x}) = -\hbar^2\frac{\vec{\nabla}^{\,2}}{2m}\psi(t,\vec{x}) \tag{3.1}$$

  Note: wave equation invariant under Galilei trafo $\vec{x}' = R\vec{x} + \vec{a}, t' = t + \Delta t$
  $\Rightarrow \psi'(t,\vec{x}) = \psi\big(t - \Delta t, R^{-1}(\vec{x} - \vec{a})\big)$ is solution if $\psi(t,\vec{x})$ is.

Relativistic generalization:

Idea: $E \to \mathrm{i}\hbar\dfrac{\partial}{\partial t}$ and $\vec{p} \to \dfrac{\hbar}{\mathrm{i}}\vec{\nabla}$ in energy-momentum relation

- $E = \dfrac{\vec{p}^{\,2}}{2m}$ $\Rightarrow$ Schrödinger equation
- $E = c\sqrt{\vec{p}^{\,2} + (mc)^2}$ $\Rightarrow$ problem with arbitrarily high spatial derivatives, no covariance !
- $E^2 = c^2\vec{p}^{\,2} + (mc^2)^2$ $\Rightarrow$ $-\dfrac{1}{c^2}\dfrac{\partial^2}{\partial t}\phi = \left(-\hat{\vec{\nabla}}^{\,2} + \underbrace{\left(\dfrac{mc}{\hbar}\right)^2}_{1\ /\ \text{(reduced Compton wave length)}}\right)\phi$

$$\overset{\hbar,c\to 1}{\Longrightarrow} \quad \left(\partial_\mu\partial^\mu + m^2\right)\phi = \left(\Box + m^2\right)\phi = 0 \qquad \textit{Klein–Gordon equation} \tag{3.2}$$

$\hookrightarrow$ ansatz as wave equation for complex wave function ϕ

Relativistic covariance:

ϕ in two different frames of reference ($x' = \Lambda x + a$): $\phi'(x') = \phi'(\Lambda x + a) = \phi(x)$

- $0 \;=\; \underbrace{\left(\Box + m^2\right)}_{\text{invariant}}\phi(x) \;=\; (\Box' + m^2)\,\phi'(x')$ $\Rightarrow$ form invariance of KG eq.
- $0 \;=\; (\Box + m^2)\,\phi(x) \;=\; (\Box + m^2)\,\phi'(\Lambda x + a)$

  $\Rightarrow$ $\phi'(x) \;=\; \phi\Big(\Lambda^{-1}(x - a)\Big)$ obeys KG eq. as well.

3.2 Solutions of the Klein–Gordon equation

Fourier ansatz in momentum space: (KG eq. = linear!)

$$\phi(x) = \phi(t, \vec{x}) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}px}\,\tilde{\phi}(p)$$

$$\hookrightarrow \left(\Box + m^2\right)\phi(x) \;=\; \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}px}\,(-p^2 + m^2)\,\tilde{\phi}(p) = 0$$

$\Rightarrow$ All $\tilde{\phi}(p)$ with $p^2 = m^2$ are solutions with $p^0 = \pm\omega_p$, where $\omega_p = +\sqrt{\vec{p}^{\,2} + m^2}$.

> Comment:
> The sign ambiguity is due to the square of E in energy–momentum relation (genuine relat. feature!).

General solution:

$$\phi(t,\vec{x}) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\,(2\pi)\,\underbrace{\delta(p^2-m^2)}_{\text{ensures } p^2=m^2}\Big[\theta(+p_0)\mathrm{e}^{-\mathrm{i}px}\,\underbrace{a(\vec{p})}\,+\theta(-p_0)\mathrm{e}^{-\mathrm{i}px}\,\underbrace{b^*(-\vec{p})}_{\text{arbitrary complex functions}}\Big]$$

$$= \underbrace{\int \frac{\mathrm{d}^4p}{(2\pi)^4}\,(2\pi)\delta(p^2-m^2)\,\theta(+p_0)}_{=\int \frac{\mathrm{d}^3p}{2\omega_p(2\pi)^3}\;\equiv\;\int \mathrm{d}\tilde{p}\;=\;\textit{invariant phase-space volume}}\left[\mathrm{e}^{-\mathrm{i}px}a(\vec{p})+\mathrm{e}^{+\mathrm{i}px}b^*(\vec{p})\right]$$

$$= \int \mathrm{d}\tilde{p}\left[\mathrm{e}^{-\mathrm{i}px}a(\vec{p})+\mathrm{e}^{+\mathrm{i}px}b^*(\vec{p})\right] \tag{3.3}$$

> Comment:
> The integral $\int \mathrm{d}^4p$ is Lorentz invariant, since $\int \mathrm{d}^4p' = \int \mathrm{d}^4p\,|\det(\Lambda)|$ with $|\det(\Lambda)|=1$.
> Moreover, $\mathrm{sgn}(p'_0)=\mathrm{sgn}(p_0)$ for $p'=\Lambda p$ and $p^2\geq 0$.
>
> Some explicit formulas:
> $\displaystyle\int \frac{\mathrm{d}^4p}{(2\pi)^4} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\int\frac{\mathrm{d}p^0}{2\pi}$,
> $\displaystyle\int \mathrm{d}p^0\,\delta(p^2-m^2) = \int \mathrm{d}p^0\,\delta\Big((p^0)^2-\vec{p}^{\,2}-m^2\Big) = \int \mathrm{d}p^0\,\frac{1}{2\omega_p}\Big[\delta\left(p^0+\omega_p\right)+\delta\left(p^0-\omega_p\right)\Big]$

Note: Negative-energy solutions b^* raise problems.

- Energy spectrum $p^0\in(-\infty,-m]\cup[m,\infty)$ not bounded from below.
  $\hookrightarrow$ Particle can emit an infinite amount of energy (by perturbations).
  $\Rightarrow$ System is unstable (no ground state) !

- Conversely, redefining $p^0\geq 0$ leads to solutions with “wrong” time-evolution phase factor $\mathrm{e}^{+\mathrm{i}p^0t}$ from non-relat. QM point of view.

- Setting $b(\vec{p})\equiv 0$ is not consistent in presence of interactions.
  $\hookrightarrow$ No solution of stability problem.

- QFT solves problem upon intepreting b^* solutions as antiparticles.
  $\hookrightarrow$ a, a^* (b^*, b) become annihilation/creation operators for (anti)particles.

3.3 Conserved current

Interpretation of ϕ as quantum mechanical wave function?

$\hookrightarrow$ Requirement: conserved "probability current" $\vec{j}$ with probability density ρ, obeying $\dot{\rho} + \vec{\nabla}\vec{j} = 0$, so that $\int \mathrm{d}^3x\, \rho(t,\vec{x}) = \text{const.}$

Recall non-relat. QM:

$\vec{j}, \rho$ derived from Schrödinger equation and its conjugate:

$$\mathrm{i}\frac{\partial}{\partial t}\psi = -\frac{1}{2m}\vec{\nabla}^2\psi\,, \qquad -\mathrm{i}\frac{\partial}{\partial t}\psi^* = -\frac{1}{2m}\vec{\nabla}^2\psi^*\,,$$

$$\Rightarrow \mathrm{i}\left[\psi^*\frac{\partial}{\partial t}\psi + \psi\frac{\partial}{\partial t}\psi^*\right] = -\frac{1}{2m}\left[\psi^*\vec{\nabla}^2\psi - \psi\vec{\nabla}^2\psi^*\right]$$

$$\Rightarrow \frac{\partial}{\partial t}\underbrace{|\psi|^2}_{=\rho} = \vec{\nabla}\cdot\underbrace{\left[\frac{\mathrm{i}}{2m}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)\right]}_{=-\vec{j}}\,. \tag{3.4}$$

Analogous manipulation with KG equation:

$$(\partial_\mu\partial^\mu + m^2)\phi = 0\,, \qquad (\partial_\mu\partial^\mu + m^2)\phi^* = 0\,,$$

$$\Rightarrow 0 = \phi^*(\partial_\mu\partial^\mu + m^2)\phi - \phi(\partial_\mu\partial^\mu + m^2)\phi^*$$

$$= \partial_\mu\underbrace{[\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*]}_{=-2m\mathrm{i}\,j^\mu}, \qquad \text{continuity equation } \checkmark \tag{3.5}$$

$\Rightarrow$ Conserved current: $\quad j^\mu = \dfrac{i}{2m}\,[\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*], \quad$ with $\partial^\mu j_\mu = 0$

$$\rho \overset{?}{=} j^0 = \frac{\mathrm{i}}{2m}\left[\phi^*\partial^0\phi - \phi\partial^0\phi^*\right] \;=\; \text{acceptable probability density ?}$$

$\hookrightarrow$ Problem: ρ can become negative, i.e. $\rho \neq$ probability density

QFT solution:

Conserved current $\propto$ $j^\mu = (\rho, \vec{j})$ interpreted as charge (ρ) and current ($\vec{j}$) density of electric (or generalized) charge.

3.4 Interpretation of the Klein–Gordon equation

Clashes between principles of non-relat. QM and KG eq. / relat. covariance:

- Special relativity: space and time should be treated on equal footing.
  QM: position is treated as an operator, time as a parameter.
- Negative energy solutions of the KG eq.
- Conserved density j^0 cannot be interpreted as probability density.
- Fields ϕ_μ transforming as 4-vectors under Lorentz transformation cannot be interpreted as wave functions as the matrices Λ for boosts are not unitary.
- Non-vanishing probability for propagation over space-like distances:

  Assume resolution of momentum $\vec{p}$: $\Delta p^i \leq mc$
  $\hookrightarrow$ localization of a particle only within $\Delta x^i \sim \dfrac{\hbar}{\Delta p^i} \gtrsim \dfrac{\hbar c}{mc^2}$
  $\hookrightarrow$ probability $\neq 0$ for propagation between space-like separated events a and b if $(x_a - x_b)^2 \sim (\hbar/mc)^2$
  $\Rightarrow$ Violation of causality due to quantum fluctuations ?

Outlook to solutions by relat. QFT:

- Field $\phi(t, \vec{x})$ satisfies the (covariant!) KG equation.
- $\vec{x}$ and t are both treated as parameters.
  $\hookrightarrow$ Elimination of the asymmetry of space and time.
- For a quantum mechanical description, the field is promoted to an operator:

  $$\phi(t, \vec{x}) \to \hat{\phi}(t, \vec{x})$$

  acting on particle states $\in$ Hilbert space:

  – action of $\hat{\phi}^\dagger(x)$ creates a particle / annihilates an antiparticle at point x;
  – action of $\hat{\phi}(x)$ creates an antiparticle / annihilates a particle at point x.

  $\Rightarrow$ QFT naturally becomes a many-particle theory.

$\hookrightarrow$ Formalization best done within Lagrangian approach to continuum mechanics

Chapter 4

Classical Field Theory

4.1 Lagrangian and Hamiltonian formalism

4.1.1 Lagrangian field theory

Recapitulation of classical mechanics of point particle

- generalized coordinates $q_i = q_i(t)$ and velocities $\dot{q}_i = \frac{\mathrm{d}q_i}{\mathrm{d}t}$
- action $S = \displaystyle\int_{t_a}^{t_b} \mathrm{d}t \, \underbrace{L(q_i, \dot{q}_i, t)}_{\text{Lagrange function}}$
- Hamilton's principle: $S =$ extremal, i.e. $\delta S = 0$, with respect to the variations

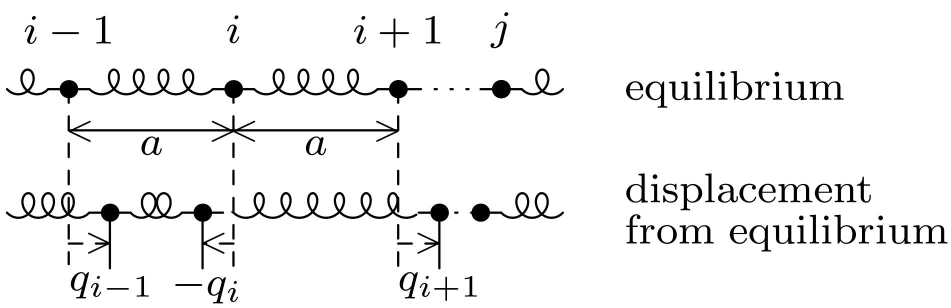
$$q_i(t) \to q_i(t) + \delta q_i(t), \quad \dot{q}_i(t) \to \dot{q}_i(t) + \delta\dot{q}_i(t), \quad \delta\dot{q}_i(t) \equiv \frac{\mathrm{d}}{\mathrm{d}t}\delta q_i(t),$$

with the boundary conditions: $\delta q_i(t_{a/b}) = 0$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \qquad \textit{Euler–Lagrange equations of motion}$$

From discrete to continuous systems:

Example (see e.g. Ref. [9]):
chain of equal mass points with mass m connected with massless uniform springs with force constant D (coupled harmonic oscillators for small q_i):



- kinetic energy: $T = \frac{1}{2}\sum_i m\dot{q}_i^2$
- potential energy: $V = \frac{1}{2}\sum_i D(q_{i+1}-q_i)^2$, $|q_{i+1}-q_i| =$ extension length
- Lagrangian: $L = T - V = \sum_i a \underbrace{\left[\frac{m}{2a}\dot{q}_i^2 - \frac{a}{2}D\left(\frac{q_{i+1}-q_i}{a}\right)^2\right]}_{=L_i}$
- Euler–Lagrange equations for coordinate q_i:

$$\underbrace{\frac{m}{a}}_{\substack{=\mu \\ = \text{mass / length}}} \ddot{q}_i - \underbrace{Da}_{=Y} \underbrace{\frac{q_{i+1}-q_i}{a}}_{\substack{=s_i \\ = \text{extension / length}}} + Da\frac{q_i - q_{i-1}}{a} = 0$$

Note:
The force / length on an elastic rod is $f = Y s$ with $Y =$ Young modulus (constant!).

- Continuum limit:

$$\text{discrete } i \to \text{continuous } x, \qquad \sum_i a \to \int \mathrm{d}x$$

$$\frac{q_{i+1}(t)-q_i(t)}{a} = \frac{q(t,x+a)-q(t,x)}{a} \xrightarrow{a\to \mathrm{d}x} \frac{\partial q}{\partial x}$$

$$\Rightarrow \quad L = \int \mathrm{d}x \underbrace{\frac{1}{2}\left[\mu\,\dot{q}(t,x)^2 - Y\left(\frac{\partial q}{\partial x}\right)^2\right]}_{=\mathcal{L}=\text{Lagrangian density}}$$

Comments:

  - $\mathcal{L}$ does not only depend on $q(x)$ and $\dot{q}(x)$, but also on $\frac{\partial q}{\partial x}$ due to nearest neighbour interaction (2nd spatial derivative $\to$ next-to-nearest neighbour interaction, etc.).
  - In more dimensions $q(t,x)$ is generalized to the field $\phi(t,\vec{x})$.

Generalization to fields:

- generalized coordinates:

$$\underset{\substack{\text{discrete} \\ \text{index } i}}{q_i(t)} \xrightarrow{\text{continuum limit}} \underset{\substack{\text{continous} \\ \text{“label” } \vec{x}}}{\phi(t,\vec{x})} = \phi(x) = \text{dyn. degree of freedom at } x^\mu = (t,\vec{x})$$

Note: ϕ may carry more indices (for spin or other d.o.f.).

- Lagrange function:
$L[\phi, \dot{\phi}] = \int \mathrm{d}^3x \quad \underbrace{\mathcal{L}(\phi, \dot{\phi}, \vec{\nabla}\phi, \dots)}_{\text{Lagrangian density, “Lagrangian”}} \quad = \text{ functional of } \phi \text{ and } \dot{\phi},$
i.e. L maps $\phi, \dot{\phi}$ to a number (dependent on t, but not on $\vec{x}$)

- action S in relativistic theories:
  - $S = S[\phi] = \int \mathrm{d}t\, L[\phi, \dot{\phi}] =$ functional of field ϕ, i.e. S maps $\phi(x)$ to a constant.
  - $S \overset{!}{=}$ Lorentz invariant (= scalar)

$$= \int \mathrm{d}t\, L[\phi, \dot{\phi}] = \underbrace{\int \mathrm{d}^4x}_{\text{Lorentz invariant}} \mathcal{L}(\phi, \dot{\phi}, \vec{\nabla}\phi, \dots)$$

$\Rightarrow \; \mathcal{L} = \mathcal{L}(\phi, \dot{\phi}, \vec{\nabla}\phi, \dots) = \mathcal{L}(\phi, \partial\phi, \dots) =$ Lorentz scalar

  - $\int \mathrm{d}^4x$ extends over complete Minkowski space with $|\phi| \to 0$ sufficiently fast for $|x^\mu| \to \infty$.
  - $S[\phi]$ is invariant under the transformation $\mathcal{L} \to \mathcal{L} + \partial^\mu F_\mu(\phi, \partial\phi, \dots)$, since surface terms vanish.
$\hookrightarrow$ $\mathcal{L}$ is unique up to partial integration.

- Hamilton's principle:
$\delta S = 0$ under variation $\phi(x) \to \phi(x) + \delta\phi(x)$ with arbitrary infinitesimal $\delta\phi(x)$ vanishing at infinity:

$$\begin{aligned} 0 = \delta S \quad &= \int \mathrm{d}^4x \left[\frac{\partial\mathcal{L}}{\partial\phi(x)}\delta\phi(x) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi(x))}\partial_\mu\delta\phi(x)\right] \\ &\overset{\text{part. int.}}{=} \int \mathrm{d}^4x \left[\frac{\partial\mathcal{L}}{\partial\phi(x)} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi(x))}\right]\delta\phi(x), \end{aligned} \tag{4.1}$$

$$\Rightarrow \quad \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0 \qquad \textit{Euler–Lagrange equations for fields} \tag{4.2}$$

- Generalization to higher derivatives: recall *variational derivative*

$$\frac{\delta\mathcal{L}}{\delta\phi} = \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} + \underbrace{\partial_\mu\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)} + \dots}_{\text{only relevant for higher-order derivatives}} \tag{4.3}$$

$\Rightarrow$ Equation of motion (EOM): $\dfrac{\delta\mathcal{L}}{\delta\phi} = 0$

Comment:
Derivation of the EOM via *functional derivative*:

$$\frac{\delta F[\phi]}{\delta \phi(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(F[\phi(y) + \epsilon \delta(x-y)] - F[\phi(y)] \right) \tag{4.4}$$

Application to action functional:

$$\begin{aligned} S[\phi] &= \int \mathrm{d}^4 y \, \mathcal{L}(\phi(y), \partial\phi(y)) && (4.5) \\ \Rightarrow \frac{\delta S[\phi]}{\delta \phi(x)} &= \int \mathrm{d}^4 y \left\{ \frac{\partial \mathcal{L}}{\partial \phi(y)} \delta(x-y) + \frac{\partial \mathcal{L}}{\partial(\partial^y_\mu \phi(y))} \, \partial^y_\mu \delta(x-y) \right\} \\ &= \frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial^x_\mu \frac{\partial \mathcal{L}}{\partial(\partial^x_\mu \phi(x))} && (4.6) \end{aligned}$$

$$\Rightarrow \text{ EOM:} \qquad \frac{\delta S[\phi]}{\delta \phi(x)} = 0 \tag{4.7}$$

Rules for functional derivatives:
F, G = functionals; a, b, c = functions

- $\dfrac{\delta}{\delta \phi(x)} \phi(y) = \delta(x-y)\,, \qquad \dfrac{\delta}{\delta \phi(x)} a(y) = 0$
- $\dfrac{\delta}{\delta \phi(x)} \left(aF[\phi] + bG[\phi] \right) = a \dfrac{\delta F[\phi]}{\delta \phi(x)} + b \dfrac{\delta G[\phi]}{\delta \phi(x)}\,,$
- $\dfrac{\delta}{\delta \phi(x)} \left(F[\phi] G[\phi] \right) = \dfrac{\delta F[\phi]}{\delta \phi(x)} G[\phi] + F[\phi] \dfrac{\delta G[\phi]}{\delta \phi(x)}\,.$

Relation between variational and functional derivative
$\hookrightarrow$ consider function $f(\phi(x))$ of field $\phi(x)$ as specific type of functional

$$\frac{\delta}{\delta \phi(x)} f\left(\phi(y)\right) = \underbrace{\frac{\delta f}{\delta \phi}\left(\phi(x)\right)}_{\substack{\text{variational derivative } \frac{\delta f}{\delta \phi} \\ \text{as function of } \phi(x)}} \delta(x-y) \equiv \frac{\delta f}{\delta \phi}\, \delta(x-y) \tag{4.8}$$

4.1.2 Hamiltonian field theory

- canonically conjugated momenta:

$$\underset{\text{point particles}}{p_i = \frac{\partial L}{\partial \dot{q}_i}} \qquad \longrightarrow \qquad \underset{\text{fields}}{\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}} \tag{4.9}$$

- Hamilton function and density:
  Replace $\dot{\phi}(x)$ by $\pi(x)$ as independent variable via Legendre transformation:

$$H(q_i, p_i) = \sum_i p_i \dot{q}_i - L \qquad \longrightarrow \qquad H = H(t) = \int \mathrm{d}^3x \; \underbrace{\mathcal{H}(\phi, \vec{\nabla}\phi, \pi)}_{\substack{\text{Hamilton density,} \\ \textit{Hamiltonian}}} \tag{4.10}$$

$$\text{with} \quad \mathcal{H}(\phi, \vec{\nabla}\phi, \pi) = \pi\dot{\phi} - \mathcal{L} \tag{4.11}$$

- Hamiltonian EOMs:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \longrightarrow \qquad \dot{\phi} = \frac{\delta \mathcal{H}}{\delta \pi}, \quad \dot{\pi} = -\frac{\delta \mathcal{H}}{\delta \phi} \tag{4.12}$$

Note: $\mathcal{H}$ and $\dot{\phi}$ depend on ϕ, $\vec{\nabla}\phi$, and π (but not on derivatives of π).

Derivation of Hamiltonian EOMs:

$$\frac{\delta \mathcal{H}}{\delta \pi} = \frac{\partial \mathcal{H}}{\partial \pi} = \dot{\phi} + \pi \frac{\partial \dot{\phi}}{\partial \pi} - \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\phi}}}_{=\pi} \frac{\partial \dot{\phi}}{\partial \pi} = \dot{\phi}\,, \tag{4.13}$$

$$\begin{aligned}
\frac{\delta \mathcal{H}}{\delta \phi} &= \frac{\partial \mathcal{H}}{\partial \phi} - \vec{\nabla} \frac{\partial \mathcal{H}}{\partial \vec{\nabla}\phi} = \pi \frac{\partial \dot{\phi}}{\partial \phi} - \frac{\partial \mathcal{L}}{\partial \phi} - \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\phi}}}_{=\pi} \frac{\partial \dot{\phi}}{\partial \phi} - \vec{\nabla} \frac{\partial \mathcal{H}}{\partial \vec{\nabla}\phi} \\
&= -\frac{\partial \mathcal{L}}{\partial \phi} - \vec{\nabla} \frac{\partial \mathcal{H}}{\partial \vec{\nabla}\phi} \overset{\text{Lag. EOM}}{=} -\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \vec{\nabla} \frac{\partial \mathcal{H}}{\partial \vec{\nabla}\phi} \\
&= -\frac{\partial}{\partial t} \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\phi}}}_{=\pi} - \vec{\nabla} \frac{\partial \mathcal{L}}{\partial \vec{\nabla}\phi} - \vec{\nabla} \frac{\partial \mathcal{H}}{\partial \vec{\nabla}\phi} = -\dot{\pi},
\end{aligned}$$

$$\text{because} \quad \left. \frac{\partial \mathcal{H}}{\partial \vec{\nabla}\phi} \right|_{\pi, \phi \text{ fixed}} = \pi \frac{\partial \dot{\phi}}{\partial \vec{\nabla}\phi} - \frac{\partial \mathcal{L}}{\partial \vec{\nabla}\phi} - \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\phi}}}_{=\pi} \frac{\partial \dot{\phi}}{\partial \vec{\nabla}\phi} = -\left. \frac{\partial \mathcal{L}}{\partial \vec{\nabla}\phi} \right|_{\dot{\phi}, \phi \text{ fixed}}. \qquad \text{q.e.d.}$$

4.2 Actions for scalar fields

Question: Which Lagrangian leads to the Klein–Gordon eq. $(\partial_\mu \partial^\mu + m^2)\,\phi = 0$, where $\phi(x) =$ real or complex scalar field ?

4.2.1 Free real scalar field

The Lagrangian

Requirements on $\mathcal{L}$:

- KG eq. is linear in ϕ; EOMs reduce $\mathcal{L}$ by one power in ϕ.
  $\hookrightarrow$ $\mathcal{L}$ is bilinear in ϕ and its derivatives $\partial_\mu \phi$, etc.
- KG involves only derivatives up to 2nd order.
  $\hookrightarrow$ $\mathcal{L}$ contains only derivatives up to 2nd order.
- $\mathcal{L} =$ Lorentz invariant.
  $\hookrightarrow$ $\mathcal{L}$ is linear combination of bilinear, Lorentz-invariant terms formed by ϕ, $\partial_\mu \phi$, etc.
  $\Rightarrow$ All possible terms: ϕ^2, $\phi\Box\phi$, $(\partial_\mu\phi)(\partial^\mu\phi)$.
  But: $\phi\Box\phi = \underbrace{\partial_\mu(\phi\partial^\mu\phi)}_{\text{irrelevant surface term}} - (\partial_\mu\phi)(\partial^\mu\phi)$ can be omitted.
- $\mathcal{L} =$ real.

$\Rightarrow$ Most general ansatz:

$$\mathcal{L} = A(\partial_\mu\phi)(\partial^\mu\phi) + B\phi^2 = Ag^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) + B\phi^2 \qquad \text{with } A, B = \text{real}$$

Determine A, B by EOM:

$$0 = \partial_\rho \frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)} - \frac{\partial\mathcal{L}}{\partial\phi} = \partial_\rho(Ag^{\mu\nu}\delta^\rho_\mu\partial_\nu\phi + Ag^{\mu\nu}\delta^\rho_\nu\partial_\mu\phi) - 2B\phi \;\; = 2A\left[\partial_\rho\partial^\rho\phi - \frac{B}{A}\phi\right] \qquad (4.14)$$

$$\hookrightarrow \;\; B/A \stackrel{!}{=} -m^2, \quad A = 1/2 \text{ (=convention, see } H \text{ below)}$$

$\Rightarrow$ Lagrangian for free scalar field:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2, \qquad \text{alternatively:} \quad \mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi \qquad (4.15)$$

Hamiltonian formalism:

- Canonically conjugated field: $\pi = \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$

  $\hookrightarrow$ Hamiltonian: $\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \dfrac{1}{2}\left[\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right]$

- Explicit solution:

$$\phi(x) = \int \mathrm{d}\tilde{p}\; \left[\mathrm{e}^{-\mathrm{i}px} a(\vec{p}) + \mathrm{e}^{+\mathrm{i}px} b^*(\vec{p})\right]\Big|_{p_0=+\sqrt{\vec{p}^2+m^2}}$$

$$\pi(x) = -\mathrm{i}\int \mathrm{d}\tilde{p}\, p_0\; \left[\mathrm{e}^{-\mathrm{i}px} a(\vec{p}) - \mathrm{e}^{+\mathrm{i}px} b^*(\vec{p})\right]\Big|_{p_0=+\sqrt{\vec{p}^2+m^2}} \tag{4.16}$$

Note: $b(\vec{p}) = a(\vec{p})$ for a real scalar field.

$\hookrightarrow$ Hamiltonian:

$$H = \int d^3x\, \mathcal{H} = \frac{1}{2}\int \mathrm{d}^3x \int \mathrm{d}\tilde{p} \int \mathrm{d}\tilde{q}\, \Big\{ \left[-p_0q_0 - \vec{p}\cdot\vec{q} + m^2\right]\left(a(\vec{p})a(\vec{q})e^{-\mathrm{i}(p+q)\cdot x} + \text{c.c.}\right)$$
$$+ \left[p_0q_0 + \vec{p}\cdot\vec{q} + m^2\right]\left(a(\vec{p})a^*(\vec{q})e^{-\mathrm{i}(p-q)\cdot x} + \text{c.c.}\right)\Big\}$$

Use identity $\displaystyle\int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} = (2\pi)^3\delta(\vec{k})$

$$= \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3\,(2p_0)^2}\Big\{\underbrace{\left[-p_0^2 + \vec{p}^2 + m^2\right]}_{=0}\left(a(\vec{p})a(-\vec{p})e^{-2\mathrm{i}p^0x^0} + \text{c.c.}\right)$$
$$+ \left[p_0^2 + \vec{p}^2 + m^2\right]\left(a(\vec{p})a^*(\vec{p}) + \text{c.c.}\right)\Big\}\Big|_{p_0=\sqrt{\vec{p}^2+m^2}}$$

$$= \int \mathrm{d}\tilde{p}\,\sqrt{\vec{p}^2+m^2}\,|a(\vec{p})|^2 \;=\; \text{const} \;>0 \quad \checkmark \tag{4.17}$$

$\Rightarrow$ Hamiltonian fulfills requirement on kinetic energy (constant, non-negative).

4.2.2 Free complex scalar field

- Complex scalar field ϕ can be decomposed: $\phi = \frac{1}{\sqrt{2}}(\phi_1 + \mathrm{i}\phi_2)$ with $\phi_1, \phi_2 = \text{real}$

$$(\Box + m^2)\phi_i = 0,\; i = 1,2 \qquad \Leftrightarrow \qquad (\Box + m^2)\phi = 0,\; (\Box + m^2)\phi^* = 0 \tag{4.18}$$

- Lagrangian (2 free real fields $\hat{=}$ 1 complex field):

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu\phi_1)(\partial^\mu\phi_1) - m^2\phi_1\phi_1\right] + \frac{1}{2}\left[(\partial_\mu\phi_2)(\partial^\mu\phi_2) - m^2\phi_2\phi_2\right]$$

$$= (\partial_\mu\phi^*)(\partial^\mu\phi) - m^2\phi^*\phi, \qquad \text{alternatively:} \quad \mathcal{L} = -\phi^*(\Box + m^2)\phi \tag{4.19}$$

- EOMs from variations of $\phi^{(*)}$:

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} - \frac{\partial \mathcal{L}}{\partial \phi^*} = \left(\partial_\mu \partial^\mu + m^2\right) \phi, \tag{4.20}$$

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = \left(\partial_\mu \partial^\mu + m^2\right) \phi^* \tag{4.21}$$

- Conjugate fields and Hamiltonian:

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^*, \qquad \pi^* = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = \dot{\phi} \tag{4.22}$$

$$\hookrightarrow \quad \mathcal{H} = \pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathcal{L} = |\pi|^2 + |\vec{\nabla}\phi|^2 + m^2|\phi|^2 \tag{4.23}$$

- Explicit solutions as in (4.16), but with $a(\vec{p}) \neq b(\vec{p})$
  $\hookrightarrow$ Hamiltonian:

$$H = ... = \int \mathrm{d}\tilde{p}\, \sqrt{\vec{p}^{\,2} + m^2}\left[a(\vec{p})a^*(\vec{p}) + b^*(\vec{p})b(\vec{p})\right] \;=\; \text{const} \; > 0 \quad \checkmark \tag{4.24}$$

4.3 Interacting fields

4.3.1 Scalar self-interactions

Extension analogous to $L = T - V$ in point mechanics: (real ϕ as example)

$$\mathcal{L} = \underbrace{\frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right)}_{\text{kinetic term}} \; \underbrace{-\frac{m^2}{2}\phi^2 - V(\phi)}_{\text{potential term}} \tag{4.25}$$

$$\text{with} \quad V = \underbrace{c_0}_{=0} + \underbrace{c_1}_{=0}\phi + c_3\phi^3 + c_4\phi^4 + \dots \tag{4.26}$$

Comments:

- $c_0 = 0$: arbitrary definition of energy offset
- $c_1 = 0$: arbitrary offset in ϕ, such that V is minimal in ground state $\phi \equiv 0$
  ($V \to +\infty$ for $|\phi| \to \infty$, otherwise system unstable. $\to$ V has minimum.)
- $c_2 = \frac{1}{2}m^2 \geq 0$: otherwise no minimum of V at $\phi \equiv 0$
- Dim. analysis: $[A]$ = unit of A, $\dim[A]$ = mass dimension of A.
  $\Rightarrow$ [Action S] $= [\hbar] = 1$, $\dim[\mathrm{d}^4x] = -4$, $\dim[\partial] = +1$ = dim. of mass
  $\Rightarrow$ $\dim[\mathcal{L}] = \dim[V] = 4$, $\dim[\phi] = 1$, $\dim[c_3] = 1$, $\dim[c_4] = 0$, $\dim[c_5] = -1$, ...

- Convenient convention:
  $c_n = g_n/\Lambda^{4-n}$ with g_n = dimensionless and Λ = common mass scale

- QFT: Theories with dim[couplings] < 0 are *non-renormalizable*,
  i.e. some observables diverge due to short-distance interactions (UV limit).

  But: Such theories can still be useful as low-energy *effective field theories*,
  where momenta $> \Lambda$ (distances $< 1/\Lambda$) are excluded.

- Non-renormalizable interactions can also involve derivatives of order > 2.

EOM and its Green function:

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi - \frac{\partial V}{\partial \phi}, \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

$\Rightarrow$ KG eq. with interaction:

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = \Box\phi + m^2\phi + \frac{\partial V}{\partial \phi}, \tag{4.27}$$

Non-linear 2nd order partial differential equation
$\hookrightarrow$ define Green function $D(x, y)$:

$$\left(\Box_x + m^2\right) D(x, y) = -\delta^4(x - y). \tag{4.28}$$

$\hookrightarrow$ integral equation equivalent to (4.27):

$$\Rightarrow \quad \phi(x) = \underbrace{\phi_0(x)}_{\substack{\text{solution of free KG eq.,} \\ (\Box + m^2)\phi_0 = 0}} + \int \mathrm{d}^4 y\, D(x, y)\, \frac{\partial V\,(\phi(y))}{\partial \phi} \tag{4.29}$$

Check:

$$(\Box_x + m^2)\phi(x) = \underbrace{(\Box_x + m^2)\phi_0(x)}_{=0} + \int \mathrm{d}^4\, y\, \underbrace{(\Box_x + m^2) D(x, y)}_{-\delta^4(x-y)}\, \frac{\partial V\,(\phi(y))}{\partial \phi} = -\frac{\partial V\,(\phi(x))}{\partial \phi}$$

Iterative solution for sufficiently weak interaction: *(perturbation theory)*

- 0th approximation: free motion

$$\phi(x) = \phi_0(x) \tag{4.30}$$

- 1st approximation: insert $\phi = \phi_0$ in r.h.s. of (4.29) $\rightarrow$ *Born approximation*

$$\phi_1(x) = \phi_0(x) + \int \mathrm{d}^4y\, D(x,y)\, \frac{\partial V(\phi_0(y))}{\partial \phi} \tag{4.31}$$

- 2nd approximation: inserting $\phi = \phi_1$ in r.h.s. of (4.29)

$$\begin{aligned}\phi_2(x) &= \phi_0(x) + \int \mathrm{d}^4y\, D(x,y)\, \frac{\partial V(\phi_1(y))}{\partial \phi} \\ &= \phi_0(x) + \int \mathrm{d}^4y\, D(x,y)\, \frac{\partial V(\phi_0(y))}{\partial \phi} \\ &\quad + \int \mathrm{d}^4y_1\, \frac{\partial^2 V(\phi_0(y_1))}{\partial \phi^2}\, D(x,y_1) \int \mathrm{d}^4y_2\, D(y_1,y_2)\, \frac{\partial V(\phi_0(y_2))}{\partial \phi} + \ldots,\end{aligned} \tag{4.32}$$

(Expansion to 2nd order in potential terms $\rightarrow$ singles out correction to Born approx.)

- n-th approximation: insert $\phi = \phi_{n-1}$ in r.h.s. of (4.29)

  Visualization of nth correction:
  free propagation between n local interactions with V at space–time points y_i.

  $\hookrightarrow$ Hope that

$$\phi_n(x) \xrightarrow[n\to\infty]{} \phi(x). \tag{4.33}$$

4.3.2 Explicit calculation of the Green function *(propagator)*

Defining Eq. (4.28) = linear, inhomogenous diff. eq.
$\hookrightarrow$ Fourier ansatz:

$$D(x,y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, D(k)\, \mathrm{e}^{-\mathrm{i}k\cdot(x-y)} \tag{4.34}$$

Note: $D(x,y) = D(x-y)$ because of translational invariance

Insertion of ansatz into (4.28):

$$\begin{aligned}(\Box_x + m^2) D(x,y) &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, D(k)(-k^2+m^2)\mathrm{e}^{-\mathrm{i}k\cdot(x-y)} \\ &\overset{!}{=} -\delta^4(x-y) = -\int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \mathrm{e}^{-\mathrm{i}k\cdot(x-y)}\end{aligned} \tag{4.35}$$

$$\begin{aligned}\Rightarrow D(k) &= \frac{1}{k^2-m^2} = \frac{1}{\left(k^0 - \sqrt{\vec{k}^{\,2}+m^2}\right)\left(k^0 + \sqrt{\vec{k}^{\,2}+m^2}\right)} \\ &= \frac{1}{2\sqrt{\vec{k}^{\,2}+m^2}}\left[\frac{1}{k^0-k_0^+} - \frac{1}{k^0-k_0^-}\right] \quad \text{with} \quad k_0^\pm = \pm\sqrt{\vec{k}^{\,2}+m^2}\end{aligned} \tag{4.36}$$

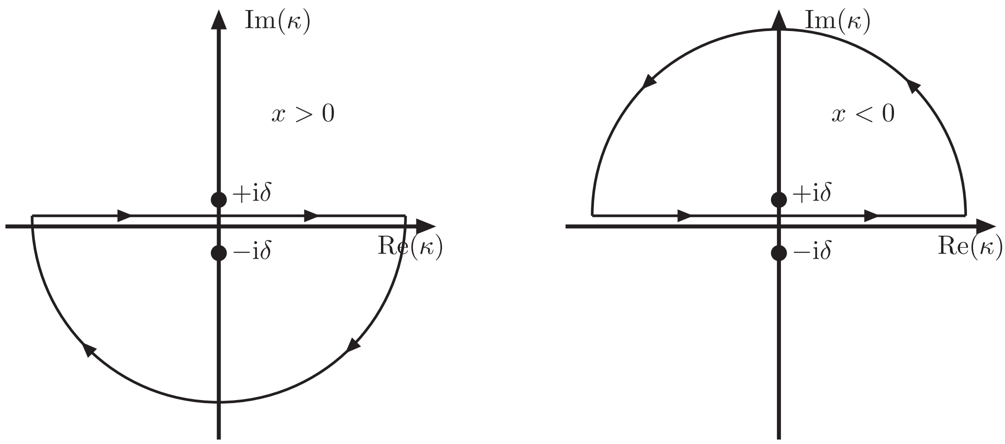
$$\Rightarrow \; D(x,y) \;=\; \int \mathrm{d}\tilde{k}\, \mathrm{e}^{\mathrm{i}\vec{k}(\vec{x}-\vec{y})} \int_{-\infty}^{\infty} \frac{\mathrm{d}k^0}{2\pi} \left[\frac{1}{k^0 - k_0^+} - \frac{1}{k^0 - k_0^-} \right] \mathrm{e}^{-\mathrm{i}k^0(x^0-y^0)} \tag{4.37}$$

Note: Prescriptions needed to resolve convergence problem near poles at $k^0 = k_0^\pm$!

Solution:
Move the poles into the complex plane by an infinitesimal shift $\mathrm{i}\delta$ ($\delta > 0$) and use identity

$$\int_{-\infty}^{\infty} \mathrm{d}\kappa \, \frac{\mathrm{e}^{-\mathrm{i}\kappa x}}{\kappa \pm \mathrm{i}\delta} = \mp 2\pi\mathrm{i}\, \theta(\pm x) \tag{4.38}$$

Comment:
Prove of identity with residue theorem:
Interpret $\int \mathrm{d}\kappa$ as line integral in complex κ-plane and close contour with half-circle of infinite radius in such a way that half-circle does not contribute:
Integrand on half-circle $\propto \exp\{\mathrm{Im}(\kappa)x\}$ $\Rightarrow$ damping for $\mathrm{Im}(\kappa)x < 0$.
$\Rightarrow$ Close contour in lower (upper) half-plane for $x > 0$ ($x < 0$).



$$\oint \mathrm{d}\kappa \, \frac{\mathrm{e}^{-\mathrm{i}\kappa x}}{\kappa \pm \mathrm{i}\delta} = \begin{cases} -2\pi\mathrm{i}, \\ 0. \end{cases} \qquad\qquad \oint \mathrm{d}\kappa \, \frac{\mathrm{e}^{-\mathrm{i}\kappa x}}{\kappa \pm \mathrm{i}\delta} = \begin{cases} 0, \\ +2\pi\mathrm{i}. \end{cases}$$

q.e.d.

Application of Eq. (4.38):

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}k^0}{2\pi} \, \frac{\mathrm{e}^{-\mathrm{i}k^0(x^0-y^0)}}{k^0 - k_0^\sigma \pm \mathrm{i}\delta} = \mp\mathrm{i}\, \mathrm{e}^{-\mathrm{i}k_0^\sigma(x^0-y^0)}\, \theta\big(\pm(x^0-y^0)\big) \quad \text{with} \quad k_0^\sigma = k_0^+ \text{ or } k_0^- \tag{4.39}$$

$\Rightarrow$ Poles at $k_0^\sigma - \mathrm{i}\delta$ correspond to forward propagation in time (contribution only for $x^0 > y^0$); poles at $k_0^\sigma + \mathrm{i}\delta$ correspond to backward propagation in time (contribution only for $y^0 > x^0$).

$\Rightarrow$ <u>4 different types of propagators:</u>

- Poles at $k_0^\pm - \mathrm{i}\delta$: *retarded propagator* $\rightarrow$ forward propagation of all modes

$$D_{\mathrm{ret}}(x,y) = -\mathrm{i}\theta(x^0-y^0)\int \mathrm{d}\tilde{k}\,\mathrm{e}^{-\mathrm{i}k(x-y)} \;+\; \text{c.c.} \;=\; \text{real} \tag{4.40}$$

  Properties: (non-trivial!)

  - Causal behaviour: $D_{\mathrm{ret}}(x,y) = 0$ for $(x-y)^2 < 0$
  - Lorentz invariance

  $\hookrightarrow$ appropriate for causal wave propagation in classical field theory

- Poles at $k_0^\pm + \mathrm{i}\delta$: *advanced propagator* $\rightarrow$ backward propagation of all modes

$$D_{\mathrm{adv}}(x,y) = D_{\mathrm{ret}}(y,x) \tag{4.41}$$

  $\hookrightarrow$ backward propagation in classical field theory

- Poles at $k_0^+ - \mathrm{i}\delta$ and $k_0^- + \mathrm{i}\delta$: *Feynman propagator*

  $\hookrightarrow$ k_0^+ with forward, k_0^- with backward propagation

$$\begin{aligned}
D_{\mathrm{F}}(x,y) &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{\mathrm{e}^{-\mathrm{i}k(x-y)}}{\left(k^0-k_0^+ + \mathrm{i}\delta\right)\left(k^0-k_0^- - \mathrm{i}\delta\right)}, \qquad k_0^\pm \mp \mathrm{i}\delta \equiv \pm\sqrt{\vec{k}^{\,2}+m^2-\mathrm{i}\epsilon} \\
&= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{\mathrm{e}^{-\mathrm{i}k(x-y)}}{k^2-m^2+\mathrm{i}\epsilon} \qquad (\text{infinitesimal } \epsilon > 0) &(4.42)\\
&= -\mathrm{i}\theta(x^0-y^0)\int \mathrm{d}\tilde{k}\,\mathrm{e}^{-\mathrm{i}k\cdot(x-y)} - \mathrm{i}\theta(y^0-x^0)\int \mathrm{d}\tilde{k}\,\mathrm{e}^{+\mathrm{i}k\cdot(x-y)} &(4.43)\\
&= D_{\mathrm{F}}(y,x) \;=\; \text{complex}
\end{aligned}$$

  Properties:

  - Lorentz invariance (obvious!), $\epsilon > 0$ acts like decay width for all modes
  - Causal behaviour non-trivial:

    $D_{\mathrm{F}}(x,y) \neq 0$ for $s^2 = (x-y)^2 < 0$ (exp. decay $\propto \mathrm{e}^{-mr}$ with $r = \sqrt{-s^2}$).

    Causality restored by independence of qm. measurements at x, y with $s^2 < 0$.
  - Propagator naturally appears in QFT:

    $x^0 > y^0$: forward propagation of particles with $k^0 > 0$;

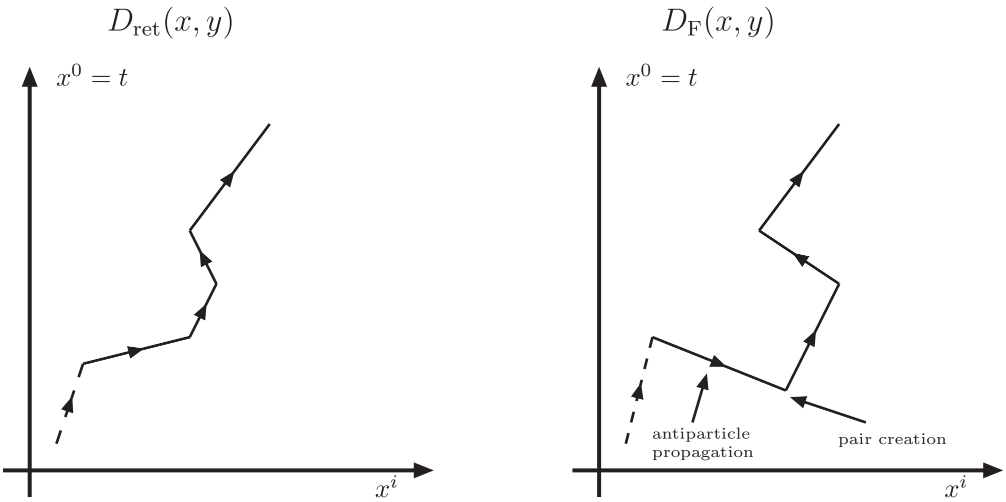
    $y^0 > x^0$: backward propagation of particles with "$k^0 < 0$"

    $\hookrightarrow$ reinterpreted as forward propagation of antiparticles

    Comment:
    No transport of information by acausal behaviour of $D_{\mathrm{F}}(x,y)$.
    Phenomenon similar to Einstein–Podolsky–Rosen paradox.

- Poles at $k_0^+ + \mathrm{i}\delta$ and $k_0^- - \mathrm{i}\delta$: Feynman propagator $D_F(x,y)^*$ for time-reversed QFT

Illustration of perturbative expansion for $\phi(x)$:



4.4 Symmetries and Noether Theorem

Noether theorem in classical mechanics:
Every continuous symmetry of a system leads to a conservation law, e.g.

- Rotational invariance $\Rightarrow$ conservation of angular momentum.
- Translational invariance $\Rightarrow$ cartesian momentum conservation.

Now: generalization to field theory.

4.4.1 Continuous symmetries

Definition:
A field theory possesses an infinitesimal continuous symmetry if a transformation

$$\phi_k \to \phi'_k \equiv \phi_k + \delta\omega_a \Delta_k^a(\phi), \tag{4.44}$$

leaves the action invariant,

$$S[\phi'] = S[\phi] \ . \tag{4.45}$$

Notation:

- $\delta\omega_a \ = \ $ infinitesimal parameters of the transformation
  $= \begin{cases} \text{const. for a } \textit{global} \text{ symmetry,} \\ \text{function}(x) \text{ for a } \textit{local} \text{ symmetry.} \end{cases}$

- $\Delta_k^a(\phi)$ = functions of all fields ϕ_k and their derivatives.
- Index a = "internal" index or Lorentz index (a may stand for multiple indices).
- Index k runs over all fields ϕ_k.

Implications for $\mathcal{L}$ and S:
If all $\phi \to 0$ sufficiently fast for $|x^\mu| \to \infty$, the invariance (4.45) of S implies that $\mathcal{L}$ can only change by a total derivative:

$$\delta\mathcal{L} \equiv \mathcal{L}(\phi') - \mathcal{L}(\phi) = \partial_\mu(K^{a,\mu}(\phi)\delta\omega_a) + \mathcal{O}(\delta\omega^2), \tag{4.46}$$

since

$$S[\phi'] = S[\phi] + \underbrace{\int_V d^4x\, \partial_\mu(K^{a,\mu}(\phi)\delta\omega_a)}_{=\text{ surface integral (Gauss!) }= 0} = S[\phi]. \tag{4.47}$$

Examples for internal symmetries:

- $U(1)$ symmetry of a complex scalar theory:
  $\mathcal{L} = (\partial_\mu\phi^*)(\partial^\mu\phi) - m^2\phi^*\phi - V(\phi^*\phi)$ = invariant under trafo

  $$\phi' = \exp\{-\mathrm{i}q\omega\}\phi, \quad \text{i.e.} \quad \Delta(\phi) = -\mathrm{i}q\phi \quad \text{with } q = \text{const.} \tag{4.48}$$

  Note: If ϕ describes an electrically charged particle, $\phi \to \phi'$ is an elmg. gauge transformation with q being the electric charge.

- $SU(N)$ symmetry of N complex scalars: $\Phi = (\phi_1, \ldots, \phi_N)^{\mathrm{T}}$
  $\mathcal{L} = (\partial_\mu\Phi)^\dagger(\partial^\mu\Phi) - m^2\Phi^\dagger\Phi - V(\Phi^\dagger\Phi)$ = invariant under trafo

  $$\Phi' = U\,\Phi, \quad \text{with} \quad U^\dagger U = \mathbf{1}, \quad \underbrace{\det(U) = +1}_{U\ =\ \text{"special"}}, \tag{4.49}$$

  $SU(N)$ = group of all special, unitary $N \times N$ matrices U.

  Exponential parametrization of U and infinitesimal transformations:

  $$\begin{aligned} U(\omega_a) &= \exp\{-\mathrm{i}gT^a\omega_a\}, \quad T^a = \textit{generators} = \text{matrices}, \ g = \text{const.} \\ U(\delta\omega_a) &= \mathbf{1} - \mathrm{i}gT^a\delta\omega_a + \ldots, \quad \text{i.e.} \quad \Delta_k^a(\phi) = -\mathrm{i}gT^a_{kl}\phi_l \end{aligned} \tag{4.50}$$

  Properties of T^a (since $U^\dagger = U^{-1}$):

  $$\begin{aligned} U(\delta\omega_a)^{-1} &= U(-\delta\omega_a) = \mathbf{1} + \mathrm{i}gT^a\delta\omega_a + \ldots \\ &= U(\delta\omega_a)^\dagger = \mathbf{1} + \mathrm{i}g(T^a)^\dagger\delta\omega_a + \ldots \quad \Rightarrow \quad T^a = (T^a)^\dagger = \text{hermitian}, \\ 1 &= \det(U) = \exp\{\mathrm{Tr}(-\mathrm{i}gT^a\omega_a)\} \quad \Rightarrow \quad \mathrm{Tr}(T^a) = T^a_{kk} = 0. \end{aligned} \tag{4.51}$$

  $\Rightarrow \quad a = 1, \ldots, n = N^2 - 1$ = # independent traceless, hermitian $N \times N$ matrices.

- $SO(N)$ symmetry of N real scalars: $\Phi = (\phi_1, \ldots, \phi_N)^{\mathrm{T}}$

  $\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^{\mathrm{T}}(\partial^\mu \Phi) - \frac{1}{2} m^2 \Phi^{\mathrm{T}} \Phi - V(\Phi^{\mathrm{T}}\Phi) =$ invariant under trafo

$$\begin{aligned} \Phi' &= R\,\Phi, \quad \text{with} \quad R^{\mathrm{T}} R = \mathbf{1}, \quad \det(R) = +1, \\ & \quad SO(N) = \text{group of all special, orthogonal } N \times N \text{ matrices } R. \end{aligned} \tag{4.52}$$

  Exponential parametrization of R and infinitesimal transformations:

$$R(\omega_a) = \exp\{-\mathrm{i} g T^a \omega_a\}, \quad \ldots \quad \Delta_k^a(\phi) = -\mathrm{i} g T^a_{kl} \phi_l. \tag{4.53}$$

  Properties of T^a (since $R^{\mathrm{T}} = R^{-1}$):

$$T^a_{kl} = -T^a_{lk} = \text{imaginary}, \quad T^a_{kk} = 0, \quad a = 1, \ldots, n = N(N-1)/2. \tag{4.54}$$

Space–time symmetries:

- Space–time translations $x^\mu \to x'^\mu = x^\mu + \omega^\mu$ with $\omega^\mu =$ const.:

$$\phi(x) \to \phi'(x) = \phi(x-\omega) = \phi(x) - \omega_\mu \partial^\mu \phi(x) + \mathcal{O}(\omega^2) \quad \Rightarrow \quad \Delta^\mu(\phi) = -\partial^\mu \phi, \tag{4.55}$$

  i.e. index a acts as Lorentz index μ.

  Transformation of the Lagrangian:

$$\delta\mathcal{L} = \mathcal{L}(x-\omega) - \mathcal{L}(x) = -\omega_\nu \partial^\nu \mathcal{L} \equiv \omega_\nu \partial_\mu K^{\mu\nu} \qquad \Rightarrow \quad K^{\mu\nu} = -g^{\mu\nu}\mathcal{L}. \tag{4.56}$$

4.4.2 Derivation of the Noether theorem

Noether theorem:

For each global symmetry of the action,

$$\phi_k \to \phi_k + \delta\omega_a \Delta_k^a(\phi)\,, \qquad \delta\mathcal{L} = \delta\omega_a \partial_\mu K^{a,\mu}, \qquad \delta\omega_a = \text{const.}, \tag{4.57}$$

there is a set of conserved *Noether currents* j_a^μ,

$$0 = \partial_\mu j_a^\mu = \partial_0 j_a^0 + \vec{\nabla}\vec{j}_a, \tag{4.58}$$

if the fields ϕ_k satisfy the EOMs.

Proof:

$$\begin{aligned} 0 &= \delta\mathcal{L} - \delta\omega_a \partial_\mu K^{a,\mu}(\phi) \\ &= \frac{\partial\mathcal{L}}{\partial\phi_k} \delta\omega_a \Delta_k^a(\phi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_k)} \delta\omega_a \partial_\mu \Delta_k^a(\phi) - \delta\omega_a \partial_\mu K^{a,\mu}(\phi) \\ &\overset{\text{EOM}}{=} \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_k)} \right] \delta\omega_a \Delta_k^a(\phi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_k)} \delta\omega_a \partial_\mu \Delta_k^a(\phi) - \delta\omega_a \partial_\mu K^{a,\mu}(\phi) \\ &= \delta\omega_a \partial_\mu \underbrace{\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_k)} \Delta_k^a(\phi) - K^{a,\mu}(\phi) \right]}_{\equiv j^{a,\mu}} \end{aligned} \tag{4.59}$$

Note: A sum over repeated labels k is implied. q.e.d.

Implications:

- Noether currents:

$$j^{a,\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \Delta_k^a(\phi) - K^{a,\mu}(\phi), \qquad \partial j^a = 0. \tag{4.60}$$

  Note: j^a only fixed up to a constant factor.

- Noether charges:

$$Q^a(t) \equiv \int \mathrm{d}^3x\, j^{a,0}(t,\vec{x}) = \int d^3x \Big[\underbrace{\pi_k}_{=\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}} \Delta_k^a(\phi) - K^{a,0}(\phi) \Big]. \tag{4.61}$$

  Charge conservation:

$$\dot{Q}^a(t) = \int_{V=\text{const.}} \mathrm{d}^3x\, \partial_0 j^{a,0}(t,\vec{x}) = -\int_V \mathrm{d}^3x\, \vec{\nabla}\cdot\vec{j}^a \underset{\text{Gauss}}{=} -\oint_{A(V)} \mathrm{d}^2\vec{A}\cdot\vec{j}^a = 0, \tag{4.62}$$

  if the currents j^a vanish sufficiently fast for $|\vec{x}| \to \infty$.

4.4.3 Internal symmetries and conserved currents

$\hookrightarrow$ Reconsider examples from Sect. 4.4.1

- $U(1)$ symmetry of the complex scalar theory:

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi - V(\phi^*\phi), \quad \phi' = \exp\{-\mathrm{i}q\omega\}\phi, \quad \Delta(\phi) = -\mathrm{i}q\phi. \tag{4.63}$$

  Noether current:

$$\begin{aligned} j^\mu &= -\mathrm{i}q \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\phi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}\phi^* \right) = -\mathrm{i}q \left[(\partial^\mu \phi^*)\phi - \phi^*(\partial^\mu \phi)\right] \qquad (4.64) \\ &= \mathrm{i}q\phi^* \overleftrightarrow{\partial^\mu} \phi \quad \text{with} \quad f(x)\overleftrightarrow{\partial_\mu} g(x) \equiv f(x)\partial_\mu g(x) - (\partial_\mu f(x))g(x). \end{aligned}$$

  Note: Result agrees (up to prefactor) with conserved current of Sect. 3.3.

  Conserved charge:

$$Q = \int \mathrm{d}^3x\, j^0(x) = \mathrm{i}q \int \mathrm{d}^3x\, \phi^* \overleftrightarrow{\partial^0} \phi = \mathrm{i}q \int \mathrm{d}^3x\, \left(\pi^*\phi^* - \phi\pi\right), \tag{4.65}$$

  with explicit solution (3.3) of free KG equation

$$\begin{aligned} Q &= -q \int \mathrm{d}\tilde{p}\, \frac{1}{2} \Big[b^*(\vec{p})b(\vec{p}) - a^*(\vec{p})a(\vec{p}) - b(\vec{p})a(-\vec{p})e^{-2\mathrm{i}p^0 t} + a^*(\vec{p})b^*(-\vec{p})e^{2\mathrm{i}p^0 t} \Big] + \text{c.c.} \\ &= q \int \mathrm{d}\tilde{p}\, \left(|a(\vec{p})|^2 - |b(\vec{p})|^2 \right). \qquad (4.66) \end{aligned}$$

  $\Rightarrow$ Positive- and negative-frequency modes carry opposite charges,
  in line with the antiparticle interpretation of the negative-frequency modes b !

- $SU(N)$ symmetry of N complex scalars:

  n Noether currents:

$$\begin{aligned} j^{a,\mu} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_k)}\Delta_k^a(\phi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_k^*)}\Delta_k^a(\phi^*) \\ &= -\mathrm{i}g\left[(\partial^\mu\phi_k^*)T_{kl}^a\phi_l - (\partial^\mu\phi_k)T_{kl}^a\phi_l^*\right], \qquad n = 1,\ldots,n. \end{aligned} \tag{4.67}$$

4.4.4 Translation invariance and energy-momentum tensor

Recall: space–time translations of fields and $\mathcal{L}$:

$$\Delta_k^\nu(\phi) = -\partial^\nu\phi_k, \qquad \delta\mathcal{L} = -\delta\omega_\nu\partial_\mu K^{\mu\nu} \quad \text{with} \quad K^{\mu\nu} = -g^{\mu\nu}\mathcal{L}. \tag{4.68}$$

$\Rightarrow$ "Current" $j \mathrel{\hat{=}}$ *energy-momentum tensor* θ: (sign of prefactor = convention)

$$\theta^{\mu\nu} \equiv -\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_k)}\Delta_k^\nu(\phi) - K^{\mu\nu}(\phi)\right] = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_k)}\partial^\nu\phi_k - g^{\mu\nu}\mathcal{L}. \tag{4.69}$$

$\Rightarrow$ Conserved "charges" Q form a 4-vector:

$$P^\mu \equiv \int \mathrm{d}^3x\, \theta^{0\mu} = \int \mathrm{d}^3x\, \left[\pi_k\partial^\mu\phi_k - g^{0\mu}\mathcal{L}\right]. \tag{4.70}$$

Individual components:

$$P^0 = \int \mathrm{d}^3x\, \left[\pi_k\dot{\phi}_k - \mathcal{L}\right] = \int \mathrm{d}^3x\, \mathcal{H} = H = \text{Hamilton function,} \tag{4.71}$$

$$\vec{P} = -\int \mathrm{d}^3x\, \pi_k\, \vec{\nabla}\phi_k = \text{ field momentum.} \tag{4.72}$$

$\Rightarrow$ "Charge" conservation $\hat{=}$ conservation of energy and momentum of the fields.

$$\dot{P}^\mu = 0. \tag{4.73}$$

> Comment:
> The "current" of Lorentz transformations forms a rank-3 tensor, and the associated "charge" is the fields' angular momentum.

Example: free complex scalar field

$\mathcal{L} = (\partial_\mu\phi^*)(\partial^\mu\phi) - m^2\phi^*\phi$.

- Energy-momentum tensor:

$$\theta^{\mu\nu} = (\partial^\mu\phi^*)(\partial^\nu\phi) + (\partial^\nu\phi^*)(\partial^\mu\phi) - g^{\mu\nu}\mathcal{L}. \tag{4.74}$$

- Hamiltonian:

$$\mathcal{H} = \pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathcal{L} = |\pi|^2 + |\vec{\nabla}\phi|^2 + m^2|\phi|^2, \qquad P^0 = \int \mathrm{d}^3x\, \mathcal{H}. \tag{4.75}$$

- Field momentum:

$$P^i = \int \mathrm{d}^3x\, \theta^{0i} = \int \mathrm{d}^3x\, (\dot{\phi}^*\, \partial^i\phi + \dot{\phi}\, \partial^i\phi^*). \tag{4.76}$$

- Insertion of plane-wave solutions yields

$$P^\mu = \int \mathrm{d}\tilde{p}\, p^\mu \left(|a(\vec{p})|^2 + |b(\vec{p})|^2\right), \tag{4.77}$$

  i.e. each mode characterized by $a(\vec{p})$, $b(\vec{p})$ carries
  energy $\propto p^0 = \sqrt{\vec{p}^{\,2} + m^2}$, momentum $\propto \vec{p}$.

Non-uniqueness issue of $\boldsymbol{\theta^{\mu\nu}}$

Possible redefinition of $\theta^{\mu\nu}$:

$$\tilde{\theta}^{\mu\nu} = \theta^{\mu\nu} + \partial_\rho\Sigma^{\rho\mu\nu}, \tag{4.78}$$

with any rank-3 tensor $\Sigma^{\rho\mu\nu} = -\Sigma^{\mu\rho\nu}$ (antisymmetry in the first two indices).

Features of $\tilde{\theta}^{\mu\nu}$:

- Conservation:

$$\partial_\mu\tilde{\theta}^{\mu\nu} = \underbrace{\partial_\mu\theta^{\mu\nu}}_{=0} + \underbrace{\partial_\mu\partial_\rho\Sigma^{\rho\mu\nu}}_{=0 \text{ by symmetry}} = 0. \tag{4.79}$$

- Field momentum:

$$\begin{aligned}\tilde{P}^\mu \quad &\equiv \quad \int_V \mathrm{d}^3x\, \tilde{\theta}^{0\mu} = P^\mu + \int_V \mathrm{d}^3x\ \underbrace{\partial_\rho\Sigma^{\rho 0\mu}}_{=\partial_i\Sigma^{i0\mu},\ \text{since } \Sigma^{00\mu}=0} \\ &\underset{\text{Gauss}}{=} \quad P^\mu + \underbrace{\oint_{A(V)} \mathrm{d}^2A^i\, \Sigma^{i0\mu}}_{=0 \text{ if } \Sigma \text{ vanishes fast enough for } |x^\mu|\to\infty} = P^\mu,\end{aligned} \tag{4.80}$$

$\Rightarrow$ Both tensors are equally suited as energy-momentum tensors.

> Comment:
> This freedom can be used to construct conserved tensors with desired properties (e.g. symmetry, gauge invariance) that are not automatically satisfied by the form directly obtained from the Noether theorem.

Chapter 5

Canonical quantization of free scalar fields

5.1 Canonical commutation relations

Consider discrete system with coordinates $q_k(t)$ and canonical conjugate momenta $p_k(t)$ and its quantization in the *Heisenberg picture*:

$q_k(t), p_k(t)$ are hermitian operators obeying

- the classical EOMs;
- Heisenberg's commutation relations.

$\hookrightarrow$ Take continuum limit $q_k(t) \;\rightarrow\; \phi(t, \vec{x})$!

> Comment:
> Heisenberg picture is more appropriate for field quantization than Schrödinger picture, because the EOM for the field (which becomes an operator) is known. Recall the connection of the two pictures by the unitary transformation for time evolution:
>
> $$\underbrace{|\psi\rangle_{\mathrm{H}}}_{\text{qm. state in H picture}} \;\equiv\; |\psi(t_0)\rangle = U^{-1}(t, t_0) \underbrace{|\psi(t)\rangle}_{\substack{\text{qm. state in S picture} \\ = t\text{-dependent}}} = \text{const.} \qquad \text{for some fixed } t_0, \tag{5.1}$$
>
> $$\underbrace{\hat{O}_{\mathrm{H}}(t)}_{\substack{\text{qm. operator} \\ \text{in H picture}}} \;\equiv\; U^{-1}(t, t_0) \underbrace{\hat{O}}_{\substack{\text{qm. operator} \\ \text{in S picture}}} U(t, t_0), \tag{5.2}$$
>
> where the time evolution operator U satisfies the differential equation
>
> $$\mathrm{i}\frac{\mathrm{d}U(t, t_0)}{\mathrm{d}t} = \hat{H} U(t, t_0) \quad \text{with} \quad U(t_0, t_0) = \mathbf{1}. \tag{5.3}$$

Quantization procedure:

Discrete system:

- Canonical variables $q_k(t)$, $p_k(t)$ obey commutators:

  $[q_k(t), p_l(t)] = \mathrm{i}\hbar\delta_{kl}$,

  $[q_k(t), q_l(t)] = 0$,

  $[p_k(t), p_l(t)] = 0$

  Note: The commutator relations only hold for equal times.

- H and L are hermitian operators obtained from classical quantities via the correspondence principle:

  $$H(q_k^{\mathrm{cl}}, p_l^{\mathrm{cl}}) \xrightarrow[\text{reordering}]{\text{unique up to}} H(q_k, p_l)$$

- The operators fulfill the EOMs:

  $$\frac{\mathrm{d}q_k(t)}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}[q_k(t), H],$$

  $$\frac{\mathrm{d}p_k(t)}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}[p_k(t), H]$$

  ($\hat{=}$ classical EOM: $\frac{1}{\mathrm{i}\hbar}[.\,,.]\hat{=}\{.\,,.\}$)

- States $|\Psi\rangle$ are time independent.

Continuous system:

- Canonical field operators $\phi_k(x)$, $\pi_k(x)$ obey

  $[\phi_k(t,\vec{x}), \pi_l(t,\vec{y})] = \mathrm{i}\hbar\delta_{kl}\delta(\vec{x}-\vec{y})$,

  $[\phi_k(t,\vec{x}), \phi_l(t,\vec{y})] = 0$,

  $[\pi_k(t,\vec{x}), \pi_l(t,\vec{y})] = 0$

  Note: The commutators only hold for equal times $t = x^0 = y^0$.

- The hermitian operators $\mathcal{H} = \pi\dot{\phi} - \mathcal{L}$ and $\mathcal{L}$ are obtained analogously:

  $$\mathcal{L}(\phi_k^{\mathrm{cl}}, \partial\phi_k^{\mathrm{cl}}) \to \mathcal{L}(\phi_k, \partial\phi_k)$$

- EOMs:

  $$\frac{\partial\phi_k(t)}{\partial t} = \frac{1}{\mathrm{i}\hbar}[\phi_k(t), H],$$

  $$\frac{\partial\pi_k(t)}{\partial t} = \frac{1}{\mathrm{i}\hbar}[\pi_k(t), H]$$

- States $|\Psi\rangle$ are time independent.

- Microcausality for space-like distances is demanded:

  $[\phi_k(x), \pi_l(y)] = 0$,

  $[\phi_k(x), \phi_l(y)] = 0$, etc.,

  for $(x-y)^2 < 0$, also for $x^0 \neq y^0$.

Comment:
In the canonical formalism Lorentz covariance is not manifest as

- ⋆ the commutator relations are imposed at equal times,
- ⋆ the Hamilton operator is not a Lorentz scalar,
- ⋆ time is singled out in the EOMs.

Observables are, however, Lorentz invariant which can be shown, for example, via the functional integral formalism.

5.2 Free Klein–Gordon field

Classical:		**QFT:**
Real KG field $\phi(x)$	$\longrightarrow$	hermitian field operator $\phi^\dagger(x) = \phi(x)$
$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2$	$\longrightarrow$	$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2$
$\Rightarrow (\Box + m^2)\phi = 0$		$\Rightarrow (\Box + m^2)\phi = 0$ (operator equation)
Complex KG fields $\phi(x)$, $\phi(x)^*$	$\longrightarrow$	non-hermitian field operators $\phi(x)$, $\phi^\dagger(x)$
$\mathcal{L} = (\partial_\mu\phi^*)(\partial^\mu\phi) - m^2\phi^*\phi$	$\longrightarrow$	$\mathcal{L} = (\partial_\mu\phi^\dagger)(\partial^\mu\phi) - m^2\phi^\dagger\phi$

Solution of KG equation:

$$\phi(x) = \int \mathrm{d}\tilde{p}\ \left[a(\vec{p})\mathrm{e}^{-\mathrm{i}px} + b^*(\vec{p})\mathrm{e}^{\mathrm{i}px}\right] \quad\longrightarrow\quad \phi(x) = \int \mathrm{d}\tilde{p}\ \left[a(\vec{p})\mathrm{e}^{-\mathrm{i}px} + b^\dagger(\vec{p})\mathrm{e}^{\mathrm{i}px}\right]$$

functions $a(\vec{p})$, $b(\vec{p})$	$\longrightarrow$	operators $a(\vec{p})$, $b(\vec{p})$
real case: $b(\vec{p}) = a(\vec{p})$		real case: $b(\vec{p}) = a(\vec{p})$

Canonical momentum:

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}^*,\ \pi^* = \frac{\partial\mathcal{L}}{\partial\dot{\phi}^*} = \dot{\phi} \quad\longrightarrow\quad \pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}^\dagger,\ \pi^\dagger = \frac{\partial\mathcal{L}}{\partial\dot{\phi}^\dagger} = \dot{\phi}$$

Meaning of the operators a, $a^\dagger$, b, $b^\dagger$?

1. Calculate a, $a^\dagger$, b, $b^\dagger$ from $\phi(x)$, $\phi^\dagger(x)$ (inverse Fourier transformation):

$$\begin{aligned}\int \mathrm{d}^3x\, \mathrm{e}^{-\mathrm{i}\vec{q}\cdot\vec{x}}\, \phi(x) &= \int \mathrm{d}\tilde{p}\ \left[a(\vec{p})\mathrm{e}^{-\mathrm{i}p^0x^0}(2\pi)^3\delta(\vec{p}-\vec{q}) + b^\dagger(\vec{p})\mathrm{e}^{\mathrm{i}p^0x^0}(2\pi)^3\delta(\vec{p}+\vec{q})\right] \\ &= \frac{1}{2q^0}\left[a(\vec{q})\mathrm{e}^{-\mathrm{i}q^0x^0} + b^\dagger(-\vec{q})\mathrm{e}^{\mathrm{i}q^0x^0}\right]_{q^0=\sqrt{\vec{q}^2+m^2}}, \end{aligned} \tag{5.4}$$

$$\int \mathrm{d}^3x\, \mathrm{e}^{-\mathrm{i}\vec{q}\cdot\vec{x}}\, \dot{\phi}(x) = -\frac{\mathrm{i}}{2}\left[a(\vec{q})\mathrm{e}^{-iq^0x^0} - b^\dagger(-\vec{q})\mathrm{e}^{iq^0x^0}\right]_{q^0=\sqrt{\vec{q}^2+m^2}}. \tag{5.5}$$

$$\Rightarrow \quad a(\vec{q}) = \mathrm{i}\int \mathrm{d}^3x \, \underbrace{\mathrm{e}^{\mathrm{i}q^0x^0}\mathrm{e}^{-\mathrm{i}\vec{q}\cdot\vec{x}}}_{=\mathrm{e}^{\mathrm{i}qx}} \left[-\mathrm{i}q^0\phi(x) + \dot{\phi}(x)\right] = \mathrm{i}\int \mathrm{d}^3x \, \mathrm{e}^{\mathrm{i}qx} \overleftrightarrow{\partial_0} \phi(x), \tag{5.6}$$

$$b(\vec{q}) = \mathrm{i}\int \mathrm{d}^3x \, \mathrm{e}^{\mathrm{i}qx} \overleftrightarrow{\partial_0} \phi^\dagger(x) \qquad \text{(derived analogously)} \tag{5.7}$$

with

$$f \overleftrightarrow{\partial_\mu} g = f(\partial_\mu g) - (\partial_\mu f) g. \tag{5.8}$$

2. Commutator relations: (choose $x^0 = y^0$)

- $$\begin{aligned}\left[a(\vec{q}), a^\dagger(\vec{p})\right] &= \int \mathrm{d}^3x \int \mathrm{d}^3y \, \mathrm{e}^{\mathrm{i}qx}\mathrm{e}^{-\mathrm{i}py} \underbrace{\left[-\mathrm{i}q^0\phi(x) + \dot{\phi}(x), \mathrm{i}p^0\phi^\dagger(y) + \dot{\phi}^\dagger(y)\right]}\Big|_{\substack{x^0 = y^0 \\ q^0 = \sqrt{\vec{q}^2 + m^2} \\ p^0 = \sqrt{\vec{p}^2 + m^2}}} \\ &\qquad = \left[-\mathrm{i}q^0\phi(x) + \pi^\dagger(x), \mathrm{i}p^0\phi^\dagger(y) + \pi(y)\right] \\ &\qquad = -\mathrm{i}q^0\left[\phi(x), \pi(y)\right] + \mathrm{i}p^0\left[\pi^\dagger(x), \phi^\dagger(y)\right] \\ &\qquad = -\mathrm{i}q^0\mathrm{i}\delta(\vec{x} - \vec{y}) + \mathrm{i}p^0(-\mathrm{i})\delta(\vec{x} - \vec{y}) \\ &= \int \mathrm{d}^3x \, \mathrm{e}^{\mathrm{i}(q-p)x}\left(q^0 + p^0\right)\Big|_{\substack{q^0 = \sqrt{\vec{q}^2 + m^2} \\ p^0 = \sqrt{\vec{p}^2 + m^2}}} \\ &= (2\pi)^3 2\sqrt{\vec{p}^2 + m^2}\,\delta(\vec{q} - \vec{p}),\end{aligned} \tag{5.9}$$

- $$\left[b(\vec{q}), b^\dagger(\vec{p})\right] = \ldots = (2\pi)^3 2\sqrt{\vec{p}^2 + m^2}\,\delta(\vec{q} - \vec{p}), \tag{5.10}$$

- $$\left[a(\vec{q}), a(\vec{p})\right] = \ldots = \left[b^\dagger(\vec{q}), b^\dagger(\vec{p})\right] = 0 \quad \text{for all other commutators.} \tag{5.11}$$

3. Energy and momentum 4-vector:

$$P^\mu = \int \mathrm{d}^3x \, \left[\pi(\partial^\mu\phi) + \pi^\dagger(\partial^\mu\phi^\dagger) - g^{\mu 0}\mathcal{L}\right] \tag{5.12}$$

> Comment:
> Ordering issue of operators solved later (*normal ordering*).

Energy:

$$H = P^0 = \int \mathrm{d}^3x \, \left[2\pi^\dagger\pi - (\partial_\mu\phi^\dagger)\partial^\mu\phi + m^2\phi^\dagger\phi\right] \tag{5.13}$$

$$= \int \mathrm{d}^3x \, \left[\pi^\dagger\pi + (\nabla\phi^\dagger)(\nabla\phi) + m^2\phi^\dagger\phi\right] \tag{5.14}$$

$$= \ldots = \int \mathrm{d}\tilde{p} \, \frac{1}{2}p^0 \left[a(\vec{p})a^\dagger(\vec{p}) + a^\dagger(\vec{p})a(\vec{p}) + b(\vec{p})b^\dagger(\vec{p}) + b^\dagger(\vec{p})b(\vec{p})\right]. \tag{5.15}$$

3-momentum:

$$\vec{P} = -\int \mathrm{d}^3x \, \left[\pi\nabla\phi + \pi^\dagger\nabla\phi^\dagger\right]$$
$$= \ldots = \int \mathrm{d}\tilde{p}\, \frac{1}{2}\vec{p}\, \left[a(\vec{p})a^\dagger(\vec{p}) + a^\dagger(\vec{p})a(\vec{p}) + b(\vec{p})b^\dagger(\vec{p}) + b^\dagger(\vec{p})b(\vec{p})\right]. \tag{5.16}$$

Commutation relations:

$$\left[H, a^\dagger(\vec{p})\right] = \frac{1}{2}\int \mathrm{d}\tilde{q}\, q^0 \Big\{ \underbrace{\left[a(\vec{q}), a^\dagger(\vec{p})\right]}_{=(2\pi)^3 2p^0\delta(\vec{q}-\vec{p})} a^\dagger(\vec{q}) + a^\dagger(\vec{q})\, \underbrace{\left[a(\vec{q}), a^\dagger(\vec{p})\right]}_{=(2\pi)^3 2p^0\delta(\vec{q}-\vec{p})} \Big\}$$
$$= p^0 a^\dagger(\vec{p}), \tag{5.17}$$

$$[H, a(\vec{p})] = -\left[H, a^\dagger(\vec{p})\right]^\dagger = -p^0 a(\vec{p}), \tag{5.18}$$

$$\left[\vec{P}, a^\dagger(\vec{p})\right] = \vec{p}a^\dagger(\vec{p}), \qquad \left[\vec{P}, a(\vec{p})\right] = -\vec{p}a(\vec{p}), \tag{5.19}$$

$$\Rightarrow \; \left[P^\mu, a^\dagger(\vec{p})\right] = p^\mu a^\dagger(\vec{p}), \qquad [P^\mu, a(\vec{p})] = -p^\mu a(\vec{p}), \tag{5.20}$$

$$\left[P^\mu, b^\dagger(\vec{p})\right] = p^\mu b^\dagger(\vec{p}), \qquad [P^\mu, b(\vec{p})] = -p^\mu b(\vec{p}). \qquad \text{(derived analogously)} \tag{5.21}$$

4. Comparison with system of independent harmonic oscillators of quantum machanics:

$$H = \sum_k \left[\frac{p_k^2}{2m} + \frac{1}{2}m\omega^2 q_k^2\right] = \sum_k \frac{\hbar\omega}{2}(a_k a_k^\dagger + a_k^\dagger a_k)$$

with shift operators a_k, $a_k^\dagger$ obeying

$$[a_k, a_l^\dagger] = \delta_{kl}, \quad [a_k, a_l] = [a_k^\dagger, a_l^\dagger] = 0 \qquad [H, a_k^\dagger] = \hbar\omega a_k^\dagger, \quad [H, a_k] = -\hbar\omega a_k.$$

Illustration for energy eigenstate $|E\rangle$:

$$H\left(a_k^\dagger|E\rangle\right) = [H, a_k^\dagger]|E\rangle + a_k^\dagger H|E\rangle = (\hbar\omega + E)a_k^\dagger|E\rangle,$$

i.e. $a_k^\dagger|E\rangle$ is energy eigenstate to energy $E + \hbar\omega$ ($a_k^\dagger$ “creates” energy $\hbar\omega$).

$\Rightarrow$ Interpretation of $a(\vec{p})$, $a^\dagger(\vec{p})$, $b(\vec{p})$, $b^\dagger(\vec{p})$ as creation and annihilation operators for field modes (=*particle excitations*):

- $a^{(\dagger)}$ and $b^{(\dagger)}$ correspond to two independent, free particle types X and $\bar{X}$, respectively, both with mass m.
- $a(\vec{p})$ / $a^\dagger(\vec{p})$ annihilates / creates particle X with energy $\hbar\omega = p^0 = \sqrt{\vec{p}^2 + m^2}$ and 3-momentum $\vec{p}$ (de Broglie momentum).

- $b(\vec{p})$ / $b^\dagger(\vec{p})$ annihilates / creates particle $\bar{X}$ with energy $\hbar\omega = p^0 = \sqrt{\vec{p}^{\,2}+m^2}$ and 3-momentum $\vec{p}$.

5. Electric current density and charge operators [cf. Eq. (4.64)]:

$$\begin{aligned} j^\mu &= -\mathrm{i}q\left[(\partial^\mu\phi^\dagger)\phi - \phi^\dagger(\partial^\mu\phi)\right] = \mathrm{i}q\phi^\dagger\overleftrightarrow{\partial^\mu}\phi, && (5.22)\\ Q &= \mathrm{i}q\int \mathrm{d}^3x\,\left(\phi^\dagger\pi^\dagger - \pi\phi\right) = \mathrm{i}q\phi^\dagger\overleftrightarrow{\partial^0}\phi && (5.23)\\ &= \ldots = q\int \mathrm{d}\tilde{p}\,\left[a^\dagger(\vec{p})a(\vec{p}) - b(\vec{p})b^\dagger(\vec{p})\right]. && (5.24)\end{aligned}$$

> Comment:
> Ordering issue of operators solved later (*normal ordering*).

$\Rightarrow$ Commutation relations:

$$\left[Q, a^\dagger(\vec{p})\right] = +qa^\dagger(\vec{p}), \qquad [Q, a(\vec{p})] = -qa(\vec{p}), \qquad (5.25)$$
$$\left[Q, b^\dagger(\vec{p})\right] = -qb^\dagger(\vec{p}), \qquad [Q, b(\vec{p})] = +qb(\vec{p}), \qquad (5.26)$$

i.e. $a^\dagger$ and b increase charge by amount q, while a and $b^\dagger$ reduce charge by amount q.

$\Rightarrow$ Particle X carries charge $+q$, particle $\bar{X}$ carries charge $-q$ ($\bar{X}$ = *antiparticle*).

Def.: *Charge conjugation* C

$$\phi^C(x) \equiv \phi^\dagger(x) = \int \mathrm{d}\tilde{p}\,\left[b(\vec{p})\mathrm{e}^{-\mathrm{i}px} + a^\dagger(\vec{p})\mathrm{e}^{\mathrm{i}px}\right], \qquad (\phi^\dagger)^C(x) \equiv \phi(x), \qquad (5.27)$$

i.e. C interchanges particle and antiparticle.

Real KG field:

- Hermitian field operator $\phi(x) = \phi^\dagger(x) = \phi^C(x)$, i.e. $a(\vec{p}) = b(\vec{p})$.

  $\Rightarrow$ $X \equiv \bar{X}$ (Particle is its own antiparticle.)

- Factor 1/2 in $\mathcal{H}$ and $\mathcal{L}$.

$$\Rightarrow\; P^\mu = \int \mathrm{d}\tilde{p}\,\frac{1}{2}p^\mu\,\left[a(\vec{p})a^\dagger(\vec{p}) + a^\dagger(\vec{p})a(\vec{p})\right] \qquad (5.28)$$

with

$$\left[P^\mu, a^\dagger(\vec{p})\right] = p^\mu a^\dagger(\vec{p}), \qquad [P^\mu, a(\vec{p})] = -p^\mu a(\vec{p}). \qquad (5.29)$$

- Electric current and charge operators: $j^\mu = \text{const.}\times\mathbf{1}$, $Q = \text{const.}\times\mathbf{1}$.

$$\Rightarrow\; \left[Q, a^\dagger(\vec{p})\right] = [Q, a(\vec{p})] = 0, \qquad (5.30)$$

i.e. particle creation / annihilation does not change overall charge.

$\Rightarrow$ Charge $q_X = 0$, X is electrically neutral.

> Comment:
> The problem with the (divergent) constant in Q is solved by *normal ordering* (=part of *renormalization process*).

5.3 Particle states and Fock space

Idea: construct Hilbert space of qm. states upon applying creation operators to ground state (analogy to qm. harmonic oscillator).

Definition: *Fock space*

- Ground state $|0\rangle$ (*vacuum*, no particle excitation):

$$|0\rangle : \quad a(\vec{p})\,|0\rangle = 0\,, \quad b(\vec{p})\,|0\rangle = 0 \quad \forall\,\vec{p}, \tag{5.31}$$

$$\langle 0| = (|0\rangle)^\dagger : \quad \langle 0|\, a^\dagger(\vec{p}) = 0\,, \quad \langle 0|\, b^\dagger(\vec{p}) = 0, \tag{5.32}$$

$$\text{Normalization:} \quad \langle 0|0\rangle = 1. \tag{5.33}$$

  Note: $|0\rangle$ exists, otherwise energy is not bounded from below.

- Excited states (particle states):

$$|X(\vec{p}_1)\rangle = a^\dagger(\vec{p}_1)\,|0\rangle \qquad \text{1 particle} \tag{5.34}$$

$$|\bar{X}(\vec{p}_1)\rangle = b^\dagger(\vec{p}_1)\,|0\rangle \qquad \text{1 antiparticle} \tag{5.35}$$

$$|X(\vec{p}_1)X(\vec{p}_2)\rangle = a^\dagger(\vec{p}_1)a^\dagger(\vec{p}_2)\,|0\rangle \qquad \text{2 particles} \tag{5.36}$$

$$|\bar{X}(\vec{p}_1)\bar{X}(\vec{p}_2)\rangle = b^\dagger(\vec{p}_1)b^\dagger(\vec{p}_2)\,|0\rangle \qquad \text{2 antiparticles} \tag{5.37}$$

$$|X(\vec{p}_1)\bar{X}(\vec{p}_2)\rangle = a^\dagger(\vec{p}_1)b^\dagger(\vec{p}_2)\,|0\rangle \qquad \text{1 particle, 1 antiparticle}$$

$$\vdots \tag{5.38}$$

$$|X(\vec{p}_1)\dots X(\vec{p}_n)\rangle = a^\dagger(\vec{p}_1)\dots a^\dagger(\vec{p}_n)\,|0\rangle \qquad n \text{ particles} \tag{5.39}$$

$$\vdots$$

- *Fock space* = Hilbert space spanned by all the particle and antiparticle states:

$$\left\{\, |0\rangle\,, |X(\vec{p}_1)\rangle\,, |\bar{X}(\vec{q}_1)\rangle\,, \dots, |X(\vec{p}_1)\dots X(\vec{p}_n)\bar{X}(\vec{q}_1)\dots\bar{X}(\vec{q}_m)\rangle\,, \dots\, \right\}$$

Properties of Fock states:

- The many-particle states are symmetric with respect to particle exchange:

$$
\begin{aligned}
|\ldots X(\vec{p}_i)\ldots X(\vec{p}_j)\ldots\rangle &= |\ldots X(\vec{p}_j)\ldots X(\vec{p}_i)\ldots\rangle \\
|\ldots \bar{X}(\vec{p}_i)\ldots \bar{X}(\vec{p}_j)\ldots\rangle &= |\ldots \bar{X}(\vec{p}_j)\ldots \bar{X}(\vec{p}_i)\ldots\rangle
\end{aligned}
\tag{5.40}
$$

  $\Rightarrow$ Particles X and $\bar{X}$ are *bosons*. (Fermionic states are antisymmetric.)

- Normalization of one-particle states:

$$
\begin{aligned}
\langle X(\vec{p})\,|X(\vec{q})\,\rangle &= \langle 0\,|a(\vec{p})a^\dagger(\vec{q})|0\,\rangle \overset{a|0\rangle=0}{=} \langle 0\,|\left[a(\vec{p}),a^\dagger(\vec{q})\right]|0\,\rangle \\
&= (2\pi)^3 2p^0\delta(\vec{p}-\vec{q})\underbrace{\langle 0|0\rangle}_{=1} = (2\pi)^3 2p^0\delta(\vec{p}-\vec{q}) = \text{Lorentz invariant},
\end{aligned}
\tag{5.41}
$$

$$
\langle \bar{X}(\vec{p})\,|\bar{X}(\vec{q})\,\rangle = \ldots = (2\pi)^3 2p^0\delta(\vec{p}-\vec{q}), \tag{5.42}
$$

$$
\langle X(\vec{p})\,|\bar{X}(\vec{q})\,\rangle = 0. \tag{5.43}
$$

  Note: $\langle X(\vec{p})\,|X(\vec{p})\,\rangle \to \infty$ for momentum eigenstates.
  $\hookrightarrow$ Wave packets needed to obtain normalized one-particle states $|\Psi\rangle$ with $\langle\Psi|\Psi\rangle = 1$.

Vacuum state and normal ordering:

Vacuum state $|0\rangle$: no particle, i.e. $\langle 0|P^\mu|0\rangle \overset{!}{=} 0$, $\langle 0|Q|0\rangle \overset{!}{=} 0$, etc.

But: These conditions are not fulfilled automatically.

$\hookrightarrow$ Enforce condition "by hand" (concept of *normal ordering*), since usually only changes in energy, momentum, etc., are measurable.

Example: vacuum energy of real scalar field

$$
\begin{aligned}
\langle 0|H|0\rangle &= \langle 0|\int \mathrm{d}\tilde{p}\,\frac{1}{2}p^0[a(\vec{p})a^\dagger(\vec{p}) + a^\dagger(\vec{p})a(\vec{p})]\,|0\rangle \\
&= \int \mathrm{d}\tilde{p}\,\frac{p^0}{2}\Big[\,\langle 0|\underbrace{[a(\vec{p}),a^\dagger(\vec{p})]}_{\substack{=(2\pi)^3 2p^0\delta(\vec{p}-\vec{p}),\\ \text{with } (2\pi)^3\delta(\vec{p}-\vec{p})\to V\,=\,\text{space volume}}} + 2\underbrace{\langle 0|a^\dagger(\vec{p})a(\vec{p})|0\rangle}_{=0}|0\rangle\Big] \\
&= \underbrace{V\int\frac{\mathrm{d}^3p}{(2\pi)^3}}_{\text{number of states in V}}\frac{p^0}{2} \;\to\; \infty
\end{aligned}
\tag{5.44}
$$

$\Rightarrow \langle H \rangle$ for a state $|f\rangle = \int \mathrm{d}\tilde{p}\, f(\vec{p})\, |X(\vec{p})\rangle$ (wave packet),

f being a square-integrable function, $\int \mathrm{d}\tilde{p}\, |f(\vec{p})|^2 < \infty$:

$$\langle f|H|f\rangle = \underbrace{\int \mathrm{d}\tilde{p}\, |f(\vec{p})|^2\, p^0}_{\text{finite, observable}} \quad + \quad \underbrace{\langle 0|H|0\rangle}_{\substack{\to\infty, \\ \text{non-observable for} \\ \text{time-independent } |0\rangle}} \tag{5.45}$$

$\hookrightarrow$ Redefinition of H upon subtracting $\langle 0|H|0\rangle$

Definition: *normal ordering*

$$: A : \equiv \left. A \right|_{\text{all annihilation operators shifted to the right}} \quad \Rightarrow \quad \langle 0| : A : |0\rangle = 0 \tag{5.46}$$

Examples: $: a(\vec{p}) a^\dagger(\vec{p}) : = \; a^\dagger(\vec{p}) a(\vec{p})$,

$: a(\vec{k}) a^\dagger(\vec{k}) a(\vec{p}) a^\dagger(\vec{p}) : = \; a^\dagger(\vec{k}) a^\dagger(\vec{p}) a(\vec{k}) a(\vec{p})$.

Redefinition of all operators, also of $\mathcal{L}$, $\mathcal{H}$, etc.:

- Definitions:

particle number density operator: $$N_X(\vec{p}) = a^\dagger(\vec{p}) a(\vec{p}), \tag{5.47}$$

particle number operator: $$N_X = \int \mathrm{d}\tilde{p}\, N_X(\vec{p}), \tag{5.48}$$

antiparticle number density operator: $$N_{\bar{X}}(\vec{p}) = b^\dagger(\vec{p}) b(\vec{p}), \tag{5.49}$$

antiparticle number operator: $$N_{\bar{X}} = \int \mathrm{d}\tilde{p}\, N_{\bar{X}}(\vec{p}). \tag{5.50}$$

- 4-momentum operator:

$$P^\mu = \int \mathrm{d}\tilde{p}\, p^\mu \left[N_X(\vec{p}) + N_{\bar{X}}(\vec{p}) \right]. \tag{5.51}$$

- electric charge operator:

$$Q = q \int \mathrm{d}\tilde{p}\, \left[N_X(\vec{p}) - N_{\bar{X}}(\vec{p}) \right]. \tag{5.52}$$

5.4 Field operator and wave function

Interpretation of field operator

Consider one-particle wave packet for particle X (charged scalar):

$$|f\rangle = \int \mathrm{d}\tilde{p}\, f(\vec{p})\,|X(\vec{p})\rangle = \int \mathrm{d}\tilde{p}\, f(\vec{p}) a^{\dagger}(\vec{p})\,|0\rangle \tag{5.53}$$

Interference with state $\phi^{\dagger}(x)\,|0\rangle$:

$$\langle 0|\phi(x)|f\rangle = \int \mathrm{d}\tilde{p}\, f(\vec{p})\,\langle 0|\phi(x)a^{\dagger}(\vec{p})|0\rangle \tag{5.54}$$

$$= \int \mathrm{d}\tilde{p}\, f(\vec{p}) \int \mathrm{d}\tilde{k}\,\mathrm{e}^{-\mathrm{i}kx}\,\underbrace{\langle 0|a(\vec{k})a^{\dagger}(\vec{p})|0\rangle}_{=(2\pi)^3(2k^0)\delta(\vec{k}-\vec{p})} \tag{5.55}$$

$$= \int \mathrm{d}\tilde{p}\,\mathrm{e}^{-\mathrm{i}px} f(\vec{p}) \equiv \varphi_f(x). \tag{5.56}$$

Compare with wave packet of non-relat. QM:

$$|\psi_f\rangle = |\psi_f(0)\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, f(\vec{p})\;|\vec{p}\rangle\,,$$

$$|\psi_f(t)\rangle = U(t,0)\,|\psi_f(0)\rangle = \exp\{-\mathrm{i}Ht\}\,|\psi_f\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, f(\vec{p})\,\mathrm{e}^{-\mathrm{i}p^0t}\;|\vec{p}\rangle\,\Big|_{p^0=\frac{\vec{p}^{\,2}}{2m}}\,,$$

$$\psi_f(t,\vec{x}) = \langle\vec{x}|\psi_f(t)\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, f(\vec{p})\,\mathrm{e}^{-\mathrm{i}p^0t}\;\underbrace{\langle\vec{x}|\vec{p}\rangle}_{=\mathrm{e}^{\mathrm{i}\vec{p}\vec{x}}} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathrm{e}^{-\mathrm{i}px}\, f(\vec{p})$$

$$= \langle\vec{x}|\exp\{-\mathrm{i}Ht\}|\psi_f\rangle \tag{5.57}$$

$\Rightarrow$ $\varphi(x)$ is analogue of one-particle wave function $\psi(t,\vec{x}) = \langle\vec{x}|\psi(t)\rangle$ in QM.

$\phi^{\dagger}(x)\,|0\rangle$ is analogue of $\exp\{+\mathrm{i}Ht\}\,|\vec{x}\rangle$ = Heisenberg state ($t=0$) corresponding to position eigenstate $|\vec{x}\rangle$ at time t, i.e. describes particle created at position $\vec{x}$ at time t.

Space–time transformations: $x \to x' = \Lambda x + a$

- Qm. states:

$$|f\rangle \to |f'\rangle = U(\Lambda,a)\,|f\rangle \quad \text{with } U = \text{unitary operator.} \tag{5.58}$$

$\hookrightarrow$ Transition amplitudes $\langle f'|g'\rangle = \langle f|U^{\dagger}U|g\rangle = \langle f|g\rangle$ = invariant.

- Field operator:

$$\phi(x') = U(\Lambda,a)\,\phi(x)\,U^{\dagger}(\Lambda,a), \tag{5.59}$$

so that scalar products $\langle f|...\phi(x)...|g\rangle = \langle f'|...\phi(x')...|g'\rangle$ = invariant.

- Wave function:

$$
\begin{aligned}
\varphi'(x') = \langle 0|\phi(x')|f'\rangle &= \underbrace{\langle 0|\,U(\Lambda,a)}_{=\langle 0| \,=\, \text{invariant}}\ \phi(x)\ \underbrace{U^\dagger(\Lambda,a)U(\Lambda,a)}_{=\mathbf{1}}\ |f\rangle \\
&= \langle 0|\phi(x)|f\rangle = \varphi(x). \qquad (5.60)\\
\Rightarrow\ \varphi'(x) &= \varphi\left(\Lambda^{-1}(x-a)\right). \qquad (5.61)
\end{aligned}
$$

5.5 Propagator and time ordering

Concept of *time ordering* of operators is very important in QFT.

Definition: *Time-ordering operator T*

$$
T\big[\phi_1(x_1)\ldots\phi_n(x_n)\big] \ \equiv\ \phi_{i_1}(x_{i_1})\phi_{i_2}(x_{i_2})\ldots\phi_{i_n}(x_{i_n}) \quad \text{for} \quad x^0_{i_1} > x^0_{i_1} > \ldots x^0_{i_n}, \qquad (5.62)
$$

i.e. “operators with earlier times are applied first”.

(Feynman) Propagator of complex scalar field (cf. Sect. 4.3.2)

$$
\mathrm{i}D_\mathrm{F}(x,y) = \langle 0\left|T\left[\phi(x)\phi^\dagger(y)\right]\right|0\rangle. \qquad (5.63)
$$

Proof:

Time-ordered product of two fields:

$$
T\left[\phi(x)\phi^\dagger(y)\right] = \phi(x)\phi^\dagger(y)\theta(x^0-y^0) + \phi^\dagger(y)\phi(x)\theta(y^0-x^0). \qquad (5.64)
$$

$$
\begin{aligned}
\Rightarrow\ \langle 0\left|T\left[\phi(x)\phi^\dagger(y)\right]\right|0\rangle &= \langle 0|T\int \mathrm{d}\tilde{p}\left[a(\vec{p})\mathrm{e}^{-ipx} + b^\dagger(\vec{p})\mathrm{e}^{ipx}\right] \\
&\quad\times \int \mathrm{d}\tilde{q}\left[a^\dagger(\vec{q})\mathrm{e}^{iqy} + b(\vec{q})\mathrm{e}^{-iqy}\right]|0\rangle \\
&= \theta(x^0-y^0)\int \mathrm{d}\tilde{p}\int \mathrm{d}\tilde{q}\ \mathrm{e}^{-\mathrm{i}(px-qy)}\ \langle 0|a(\vec{p})a^\dagger(\vec{q})|0\rangle \\
&\quad + \theta(y^0-x^0)\int \mathrm{d}\tilde{p}\int \mathrm{d}\tilde{q}\ \mathrm{e}^{\mathrm{i}(px-qy)}\ \langle 0|b(\vec{q})b^\dagger(\vec{p})|0\rangle
\end{aligned}
$$

- $x^0 > y^0$: particle creation at y, propagation to x, annihilation at x
- $x^0 < y^0$: antiparticle creation at x, propagation to y, annihilation at y
- Note: identical charge flow from y to x in both cases

$$
= \theta(x^0-y^0)\int \mathrm{d}\tilde{p}\ \mathrm{e}^{-\mathrm{i}p(x-y)} + \theta(y^0-x^0)\int \mathrm{d}\tilde{p}\ \mathrm{e}^{\mathrm{i}p(x-y)}
$$

$$= \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{e}^{-\mathrm{i}p(x-y)}}{p^2 - m^2 + \mathrm{i}\epsilon} \qquad \text{[see Eq. (4.42)]}$$

$$= \mathrm{i}D_\mathrm{F}(x, y) \tag{5.65}$$

q.e.d.

Chapter 6

Interacting scalar fields and scattering theory

QFT with interacting fields
$\hookrightarrow$ framework for formulating theories of fundamental interactions

But: evaluation extremely complicated
$\hookrightarrow$ systematic approximative methods needed

- Exact solutions: only for some lower-dimensional models
- *Perturbation theory:* expansion in small coupling constants g
  $\hookrightarrow$ most useful for scattering problems
- *Lattice calculations:* numerical simulations by discretizing space–time
  $\hookrightarrow$ useful for static problems (e.g. for bound states in strong interaction)

6.1 Asymptotic states and S-matrix

Asymptotic states in particle scattering

Scattering process:

$$|i\rangle \to |f\rangle \tag{6.1}$$

with

$|i\rangle$ = prepared momentum eigenstate before scattering,
evolving into a complicated mixed many-particle state $|i'\rangle$ after scattering,
$|f\rangle$ = specific final state that is contained in $|i'\rangle$ after scattering.

Relevant cases:

- 2-particle scattering: $|i\rangle = |\vec{k}_1, \vec{k}_2\rangle$
- particle decay: $|i\rangle = |\vec{k}_1\rangle$

Technical description:

- interaction in finite time interval $[-T, T]$ with $T \gg$ any relevant time scale
- initial state $|i\rangle_{\text{in}}$: $t \to -\infty$ $(t < -T)$, no interaction
- final state $|f\rangle_{\text{out}}$: $t \to +\infty$ $(t > +T)$, no interaction

Subtleties:

- Behaviour of $\phi(x)$ for $t = x^0 \to \mp\infty$:

$$\phi(x) \quad \underset{t\to\mp\infty}{\widetilde{\qquad}} \quad \underbrace{Z^{1/2}}_{\substack{\textit{wave-function}\\ \textit{renormalization}\\ \textit{constant}}} \phi_{\text{in/out}}(x), \tag{6.2}$$

  where the asymptotics holds in the "weak" sense (for matrix elements only).

  Origin of Z:
  $\phi(x)$ and "free" fields $\phi_{\text{in/out}}(x)$ are canonically normalized (commutators!), but

  - Free fields only have non-vanishing matrix elements with one-particle states:

$$\langle 0|\phi_{\text{in}}(x)|\vec{k}\rangle_{\text{in}} = \mathrm{e}^{-\mathrm{i}kx}. \tag{6.3}$$

  - Interacting fields interfere also with multiparticle states:

$$\langle 0|\phi(x)|\vec{k}_1 \ldots \vec{k}_n\rangle \neq 0. \tag{6.4}$$

  - Relation between wave functions:

$$\langle 0|\phi(x)|\vec{k}\rangle \underset{t\to\mp\infty}{\widetilde{\qquad}} Z^{1/2}\,\langle 0|\phi_{\text{in/out}}(x)|\vec{k}\rangle \tag{6.5}$$

  Comment:
  Z can be calculated from the vacuum expectation value of $[\phi[x), \phi(y)]$ (see e.g. [2, 3]). Under some assumption (finiteness), one can show that $0 < Z < 1$.

- Interaction changes mass value (*mass renormalization*):
  Asymptotic fields satisfy the free KG equation,

$$(\Box + \bar{m}^2)\phi_{\text{in/out}}(x) = 0, \tag{6.6}$$

  but mass $\bar{m} \neq m =$ mass in original Lagrangian.

- Note: in lowest order of perturbation theory: $Z = 1$ and $\bar{m} = m$.
  Higher orders deserve care, i.e. a proper *renormalization*.

Scattering operator, "*S*-matrix"

Initial and final states form orthonormal bases (ONBs) of two isomorphic Fock spaces.

- "in" states:
  $\{|\alpha\rangle_{\text{in}}\}$ = ONB of momentum eigenstates with definite particle number.
  Initial state $|i\rangle_{\text{in}} \in \{|\alpha\rangle_{\text{in}}\}$ *before scattering*
  $\hookrightarrow$ evolves into $S\,|i\rangle_{\text{in}}$ *after scattering* with unitary *scattering operator* S.
  Note: $S\,|i\rangle_{\text{in}}$ = complicated superposition of states $|\alpha\rangle_{\text{in}}$.
  Test final state $|f\rangle_{\text{in}} \in \{|\alpha\rangle_{\text{in}}\}$ for time *after scattering*
  $\hookrightarrow$ to be projected onto $S\,|i\rangle_{\text{in}}$ to measure probability to get $|f\rangle_{\text{in}}$ in scattering.
- "out" states:
  $|\alpha\rangle_{\text{out}} \equiv S^\dagger\,|\alpha\rangle_{\text{in}}$, i.e. $|\alpha\rangle_{\text{out}}$ scatters into $S\,|\alpha\rangle_{\text{out}} = |\alpha\rangle_{\text{in}}$.
  $|\alpha\rangle_{\text{out}}$ = complicated superposition of states $|\beta\rangle_{\text{in}}$, but $\{|\alpha\rangle_{\text{out}}\}$ = ONB.
  $\Rightarrow$ $|f\rangle_{\text{out}}$ = initial state *before scattering* that evolves into $|f\rangle_{\text{in}}$.
- Vacuum states can be identified:
$$S\,|0\rangle_{\text{out}} \;=\; |0\rangle_{\text{in}} \;\equiv\; |0\rangle_{\text{out}} \;\equiv\; |0\rangle\,. \tag{6.7}$$
- Probabilities for qm. transitions $|i\rangle \to |f\rangle$ are proportional to $|S_{fi}|^2$ where
$$S_{fi} = {}_{\text{in}}\langle f|S|i\rangle_{\text{in}} = {}_{\text{out}}\langle f|i\rangle_{\text{in}} = {}_{\text{out}}\langle f|S|i\rangle_{\text{out}}\,. \tag{6.8}$$
- Asymptotic field operators:
$$\phi_{\text{in}}(x) = S\,\phi_{\text{out}}(x)\,S^\dagger, \tag{6.9}$$
  so that
$${}_{\text{in}}\langle\alpha|\phi_{\text{in}}|\beta\rangle_{\text{in}} = {}_{\text{in}}\langle\alpha|S\phi_{\text{out}}S^\dagger|\beta\rangle_{\text{in}} = {}_{\text{out}}\langle\alpha|\phi_{\text{out}}|\beta\rangle_{\text{out}}\,, \quad \text{etc.} \tag{6.10}$$
- Poincaré invariance of matrix elements requires:
$$S = U(\Lambda,a)\,S\,U^\dagger(\Lambda,a), \tag{6.11}$$

Aim: perturbative expansion for S_{fi}
$\hookrightarrow$ derive relation between S, time evolution, and H

6.2 Perturbation Theory

Recapitulation of qm. time evolution pictures:

1. **Schrödinger picture:**

   States carry time evolution, described by the *time evolution operator* U:

$$|\psi(t)\rangle = U(t,t_0)\,|\psi(t_0)\rangle\,, \tag{6.12}$$

$$\mathrm{i}\frac{\mathrm{d}U(t,t_0)}{\mathrm{d}t} = H(t)U(t,t_0), \qquad U(t_0,t_0)=1. \tag{6.13}$$

   Properties of U:

   - unitarity:
$$U(t,t_0)=U^{-1}(t_0,t)=U^\dagger(t_0,t), \tag{6.14}$$
   - group composition law:
$$U(t,t')U(t',t_0)=U(t,t_0). \tag{6.15}$$

   Iterative solution:

$$\begin{aligned}
U(t,t_0) &= 1-\mathrm{i}\int_{t_0}^{t}\mathrm{d}t'\,H(t')U(t',t_0)\\
&= 1-\mathrm{i}\int_{t_0}^{t}\mathrm{d}t'\,H(t')+(-\mathrm{i})^2\int_{t_0}^{t}\mathrm{d}t'\int_{t_0}^{t'}\mathrm{d}t''\,H(t')H(t'')+\dots\\
&= 1-\mathrm{i}\int_{t_0}^{t}\mathrm{d}t'\,H(t')+\frac{(-\mathrm{i})^2}{2}\int_{t_0}^{t}\mathrm{d}t'\left[\int_{t_0}^{t'}\mathrm{d}t''\,H(t')H(t'')+\underbrace{\int_{t'}^{t}\mathrm{d}t''\,H(t'')H(t')}_{\text{interchanged integration variables } t'\leftrightarrow t''}\right]\\
&\quad +\dots\\
&= 1-\mathrm{i}\int_{t_0}^{t}\mathrm{d}t'\,H(t')+\frac{1}{2}(-\mathrm{i})^2\int_{t_0}^{t}\mathrm{d}t'\int_{t_0}^{t}\mathrm{d}t''\,T[H(t')\,H(t'')]+\dots\\
&\equiv T\exp\left[-\mathrm{i}\int_{t_0}^{t}\mathrm{d}t' H(t')\right].
\end{aligned}$$

2. **Heisenberg picture:**

   States are time-independent and tied to the S picture at some time t_0:

$$|\psi\rangle_{\mathrm{H}}=|\psi(t_0)\rangle=U(t_0,t)\,|\psi(t)\rangle\,. \tag{6.16}$$

   Operators are transformed in such a way that matrix elements remain the same:

$$O_{\mathrm{H}}(t)=U^\dagger(t,t_0)\,O(t)\,U(t,t_0). \tag{6.17}$$

$\Rightarrow$ EOM for operators:

$$\begin{aligned} \mathrm{i}\frac{\mathrm{d}O_{\mathrm{H}}(t)}{\mathrm{d}t} &= \mathrm{i}\left(\dot{U}^\dagger O(t) U + U^\dagger O(t)\dot{U}\right) + U^\dagger \mathrm{i}\frac{\partial O(t)}{\partial t} U \\ &= [O_{\mathrm{H}}(t), H_{\mathrm{H}}(t)] + \mathrm{i}\left(\frac{\partial O(t)}{\partial t}\right)_{\mathrm{H}} \end{aligned} \tag{6.18}$$

with $H_{\mathrm{H}}(t) = U^\dagger(t,t_0)\, H(t)\, U(t,t_0)$.

3. **Interaction picture:**

- Hamilton operator split into a free and an interacting part,

$$H(t) = H_0(t) + H_{\mathrm{int}}(t) \qquad \text{(spectrum of } H_0(t) \text{ known)} \tag{6.19}$$

- Time evolution from H_0 removed from states (as in H picture):

$$|\psi(t)\rangle_{\mathrm{I}} = U_0^\dagger(t,t_0)\,|\psi(t)\rangle\ , \qquad O_{\mathrm{I}}(t) = U_0^\dagger(t,t_0)\, O\, U_0(t,t_0) \tag{6.20}$$

  with the time evolution operator U_0 of H_0,

$$\mathrm{i}\frac{\mathrm{d}U_0(t,t_0)}{\mathrm{d}t} = H_0(t) U_0(t,t_0). \tag{6.21}$$

- EOM of the states:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\,|\psi(t)\rangle_{\mathrm{I}} = H_{\mathrm{I}}(t)\,|\psi(t)\rangle_{\mathrm{I}} \qquad \text{with} \quad H_{\mathrm{I}}(t) = U_0^\dagger(t,t_0) H_{\mathrm{int}}(t) U_0(t,t_0)\ , \tag{6.22}$$

  i.e. states evolve in time with the interaction Hamiltonian H_{I}.

- EOM of the operators:

$$\mathrm{i}\frac{\mathrm{d}O_{\mathrm{I}}(t)}{\mathrm{d}t} = [O_{\mathrm{I}}(t), H_{0,\mathrm{I}}(t)] + \mathrm{i}\left(\frac{\partial O(t)}{\partial t}\right)_{\mathrm{I}} \quad \text{with} \quad H_{0,\mathrm{I}}(t) = U_0^\dagger(t,t_0) H_0(t) U_0(t,t_0), \tag{6.23}$$

  i.e. operators evolve in time with the free Hamiltonian $H_{0,\mathrm{I}}$.

- Time evolution operator U_{I} in the I picture:

$$|\psi(t)\rangle_{\mathrm{I}} = U_0^\dagger(t,t_0) U(t,t_0)\,|\psi\rangle_{\mathrm{H}} \equiv U_{\mathrm{I}}(t,t_0)\,|\psi\rangle_{\mathrm{H}}\ , \tag{6.24}$$

$$O_{\mathrm{I}}(t) = U_{\mathrm{I}}(t,t_0)\, O_{\mathrm{H}}(t)\, U_{\mathrm{I}}^\dagger(t,t_0)\,, \tag{6.25}$$

$$\mathrm{i}\frac{\mathrm{d}U_{\mathrm{I}}(t,t_0)}{\mathrm{d}t} = U_0^\dagger(t,t_0) H_{\mathrm{int}}(t) U(t,t_0) = H_{\mathrm{I}}(t) U_{\mathrm{I}}(t,t_0). \tag{6.26}$$

  Formal solution:

$$U_{\mathrm{I}}(t,t_0) = T\exp\left[-\mathrm{i}\int_{t_0}^{t} \mathrm{d}t'\, H_{\mathrm{I}}(t')\right]. \tag{6.27}$$

Application to scattering in QFT:

> Comment:
> Here we assume $Z = 1$ and $\bar{m} = m$, which is sufficient for the lowest perturbative order and, thus, for this lecture.

- Initial states: take $t, t_0 < -T$ and subsequently $-T \to -\infty$.
  - ◇ H picture and I picture are identical (no interaction yet).
  - ◇ $|\psi(t)\rangle_{\mathrm{I}} = |\psi(t_0)\rangle_{\mathrm{H}} = |i\rangle_{\mathrm{in}} = \text{const.}$
  - ◇ $U_{\mathrm{I}}(t, t_0) = \mathbf{1}, \quad \phi_{\mathrm{in}}(x) = \phi(x)$.
- Interaction period: $t_0 < -T < t < T$.
  - ◇ $|\psi(t)\rangle_{\mathrm{I}} = U_{\mathrm{I}}(t, t_0)\,|i\rangle_{\mathrm{in}} = U_{\mathrm{I}}(t, -T)\,|i\rangle_{\mathrm{in}}$.
  - ◇ $\underbrace{\phi_{\mathrm{in}}(x)}_{\text{I picture}} = U_{\mathrm{I}}(t, -T)\,\underbrace{\phi(x)}_{\text{H picture}}\,U_{\mathrm{I}}^{\dagger}(t, -T)$.
- Final states: $t_0 < -T < T < t$.
  - ◇ $|\psi(t)\rangle_{\mathrm{I}} = U_{\mathrm{I}}(t, t_0)\,|i\rangle_{\mathrm{in}} = U_{\mathrm{I}}(T, -T)\,|i\rangle_{\mathrm{in}} \xrightarrow{T\to\infty} S\,|i\rangle_{\mathrm{in}}$, where
  $$S \equiv U_{\mathrm{I}}(\infty, -\infty). \tag{6.28}$$
  - ◇ $\phi_{\mathrm{in}}(x) = U_{\mathrm{I}}(t, -T)\,\phi(x)\,U_{\mathrm{I}}^{\dagger}(t, -T) = U_{\mathrm{I}}(T, -T)\,\phi_{\mathrm{out}}(x)\,U_{\mathrm{I}}^{\dagger}(T, -T)$
  $\xrightarrow{T\to\infty} S\,\phi_{\mathrm{out}}(x)\,S^{\dagger}$.
  - ◇ $|f(t)\rangle_{\mathrm{I}} \equiv U_{\mathrm{I}}(t, +T)\,|f\rangle_{\mathrm{in}} =$ test final state, where $|f\rangle_{\mathrm{in}}$ typically is some measured free momentum eigenstate.
- S-matrix element:
$$\begin{aligned} {}_{\mathrm{I}}\langle f(t)|\psi(t)\rangle_{\mathrm{I}} &= {}_{\mathrm{in}}\langle f|\,\underbrace{U_{\mathrm{I}}^{\dagger}(t, T)U_{\mathrm{I}}(t, -T)}_{=U_{\mathrm{I}}(T,-T)\to S}\,|i\rangle_{\mathrm{in}} \\ &\longrightarrow {}_{\mathrm{in}}\langle f|S|i\rangle_{\mathrm{in}} = {}_{\mathrm{out}}\langle f|i\rangle_{\mathrm{in}} = S_{fi}. \end{aligned} \tag{6.29}$$

  Note: $|i\rangle_{\mathrm{in}}$ and $|f\rangle_{\mathrm{out}}$ are Heisenberg states corresponding to $t \to -\infty$.
- Operators in the I picture:
  - ◇ Field operators (for all times t, see above) for free particle propagation:
  $$\phi_{\mathrm{in}}(x) = U_{\mathrm{I}}(t)\,\phi(x)\,U_{\mathrm{I}}^{\dagger}(t). \tag{6.30}$$

◇ Hamiltonian (needed for time evolution):

$$
\begin{aligned}
H_{\mathrm{I}}(t) &= U_{\mathrm{I}}(t) H_{\mathrm{int}}(t) U_{\mathrm{I}}^{\dagger}(t), \qquad U_{\mathrm{I}}(t) \equiv U_{\mathrm{I}}(t,-\infty) \\
&= \int \mathrm{d}^3x \, U_{\mathrm{I}}(t) \, \mathcal{H}_{\mathrm{int}}\big(\phi(x), \pi(x)\big) \, U_{\mathrm{I}}^{\dagger}(t). \\
&= \int \mathrm{d}^3x \, \mathcal{H}_{\mathrm{int}}\big(\phi_{\mathrm{in}}(x), \pi_{\mathrm{in}}(x)\big). \qquad (6.31)
\end{aligned}
$$

Example: scalar field theory with interaction potential, $\mathcal{L}_{\mathrm{int}}(\phi) = -V(\phi)$

$$
H_{\mathrm{I}}(t) = \int \mathrm{d}^3x \, U_{\mathrm{I}}(t) \, \mathcal{H}_{\mathrm{int}}\big(\phi(x)\big) \, U_{\mathrm{I}}^{\dagger}(t) = \int \mathrm{d}^3x \, \mathcal{H}_{\mathrm{int}}\big(\phi_{\mathrm{in}}(x)\big) \qquad (6.32)
$$

with

$$
\mathcal{H}_{\mathrm{int}}(\phi_{\mathrm{in}}) = V(\phi_{\mathrm{in}}) = -\mathcal{L}_{\mathrm{int}}(\phi_{\mathrm{in}}). \qquad (6.33)
$$

Comment:
This transition is more complicated if the interaction $V(\phi)$ involves derivatives, i.e. if $\mathcal{H}_{\mathrm{int}}$ involves canonical momenta. Then, in general, $\mathcal{H}_{\mathrm{int}}(\phi_{\mathrm{I}}) \neq -\mathcal{L}_{\mathrm{int}}(\phi_{\mathrm{I}})$, as e.g. in scalar QED.

- Perturbative expansion of S-matrix: $\quad (|i\rangle \equiv |i\rangle_{\mathrm{in}}, \, |f\rangle \equiv |f\rangle_{\mathrm{in}})$

$$
\begin{aligned}
S_{fi} &= \langle f| \, T \exp\left[-\mathrm{i} \int \mathrm{d}^4x \, \mathcal{H}_{\mathrm{int}}\big(\phi_{\mathrm{in}}(x)\big)\right] |i\rangle \qquad (6.34) \\
&= \langle f|i\rangle - \mathrm{i} \int \mathrm{d}^4x \, \langle f|\mathcal{H}_{\mathrm{int}}\big(\phi_{\mathrm{in}}(x)\big)|i\rangle \qquad (6.35) \\
&\quad + \sum_{n=2}^{\infty} \frac{(-\mathrm{i})^n}{n!} \left(\prod_{j=1}^{n} \int \mathrm{d}^4x_j\right) \langle f| \, T \left[\mathcal{H}_{\mathrm{int}}\big(\phi_{\mathrm{in}}(x_1)\big) \ldots \mathcal{H}_{\mathrm{int}}\big(\phi_{\mathrm{in}}(x_n)\big)\right] |i\rangle \, .
\end{aligned}
$$

6.3 Feynman diagrams

6.3.1 Wick's theorem

Task: Computation of S-matrix elements, i.e. of

$$\langle f|\, T[\mathcal{H}_{\mathrm{int}}(x_1) \ldots \mathcal{H}_{\mathrm{int}}(x_n)]\, |i\rangle\,, \tag{6.36}$$

where $\mathcal{H}_{\mathrm{int}}(x) = :\phi_{\mathrm{in}}(x)\phi^\dagger_{\mathrm{in}}(x)\cdots:$ with *free* "in"-fields ϕ_{in}, etc.

$\Rightarrow$ Translate time-ordered products into normal-ordered products,
so that $\langle f|a^\dagger(p)...a(k)|i\rangle$ can be evaluated explicitly.

Example used for illustration: one real scalar field ϕ with

$$\mathcal{H}_{\mathrm{int}}(x) = -\frac{g}{3!} :\phi^3(x): \,. \tag{6.37}$$

Note: Subscript "in" suppressed in the following.

Definition: *Contraction* of free, bosonic real field operators:

$$:\cdots \overbracket{\phi_i \cdots \phi_j} \cdots: = \langle 0|\, T[\phi_i \phi_j]\, |0\rangle \cdot \; :\ldots \phi_{i-1}\phi_{i+1} \cdots \phi_{j-1}\phi_{j+1} \cdots: \tag{6.38}$$

Example:

$$\overbracket{\phi(x)\phi(y)} \;=\; \underbrace{\langle 0|\, T[\phi(x)\phi(y)]\, |0\rangle}_{=\text{ complex number}} \;=\; \mathrm{i}D_{\mathrm{F}}(x,y), \qquad \text{Feynman propagator} \tag{6.39}$$

Identity for time-ordered product of two field operators (see Exercise 6.2):

$$\begin{aligned} T\left[\phi(x)\phi(y)\right] &= :\phi(x)\phi(y): \;+ \langle 0\,|T\left[\phi(x)\phi(y)\right]|\,0\rangle \\ &= :\phi(x)\phi(y): \;+\; \overbracket{\phi(x)\phi(y)}. \end{aligned} \tag{6.40}$$

General case of an arbitrary number of free, bosonic, real field operators ruled by

Wick's theorem: (Discussion in Exercise 7.1)

$$\begin{aligned} T[\phi_1 \cdots \phi_n] \;&=\; :\phi_1 \cdots \phi_n: \;+ \sum_{\text{pairs } ij} :\phi_1 \cdots \overbracket{\phi_i \cdots \phi_j} \cdots \phi_n: \\ &\quad + \sum_{\text{double pairs } ij,kl} :\phi_1 \cdots \contraction{}{\phi_i}{\cdots \phi_k \cdots}{\phi_j}\contraction[2ex]{\phi_i \cdots}{\phi_k}{\cdots \phi_j \cdots}{\phi_l}\phi_i \cdots \phi_k \cdots \phi_j \cdots \phi_l \cdots \phi_n: \;+\; \ldots \end{aligned} \tag{6.41}$$

Extension: If some of the fields in the argument of the T-product in (6.41) are within the same normal-ordered product, they will not be contracted.

Examples:

- four fields:

$$\begin{aligned} T[\phi_1 \cdots \phi_4] = &: \phi_1 \cdots \phi_4 : + : \phi_1\phi_2 : \overbracket{\phi_3\phi_4} + : \phi_1\phi_3 : \overbracket{\phi_2\phi_4} + : \phi_1\phi_4 : \overbracket{\phi_2\phi_3} \\ &+ : \phi_2\phi_3 : \overbracket{\phi_1\phi_4} + : \phi_2\phi_4 : \overbracket{\phi_1\phi_3} + : \phi_3\phi_4 : \overbracket{\phi_1\phi_2} \\ &+ \overbracket{\phi_1\phi_2}\overbracket{\phi_3\phi_4} + \overbracket{\phi_1\phi_3}\overbracket{\phi_2\phi_4} + \overbracket{\phi_1\phi_4}\overbracket{\phi_2\phi_3}, \end{aligned} \tag{6.42}$$

- pair of normal-ordered products:

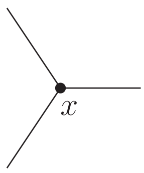
$$\begin{aligned} T[: \phi_1\phi_2 : : \phi_3\phi_4 :] = &: \phi_1 \cdots \phi_4 : + : \phi_1\phi_3 : \overbracket{\phi_2\phi_4} + : \phi_1\phi_4 : \overbracket{\phi_2\phi_3} \\ &+ : \phi_2\phi_3 : \overbracket{\phi_1\phi_4} + : \phi_2\phi_4 : \overbracket{\phi_1\phi_3} + \overbracket{\phi_1\phi_3}\overbracket{\phi_2\phi_4} + \overbracket{\phi_1\phi_4}\overbracket{\phi_2\phi_3}. \end{aligned} \tag{6.43}$$

6.3.2 Feynman rules for the S-operator

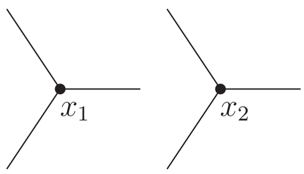
Application of Wick theorem expands S operator in terms of propagators and normal-ordered fields:

$$S = U_{\mathrm{I}}(\infty, -\infty) = T\exp\left[\int \mathrm{d}^4x \, \frac{\mathrm{i}g}{3!} : \phi^3(x) : \right]$$

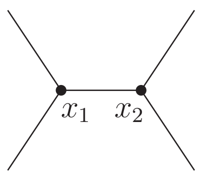
$$= \mathbf{1} \; + \; \frac{\mathrm{i}g}{3!}\int \mathrm{d}^4x \; : \phi^3(x) :$$

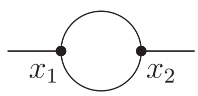


$$+ \; \frac{1}{2}\left(\frac{\mathrm{i}g}{3!}\right)^2 \int \mathrm{d}^4x_1 \, \mathrm{d}^4x_2 \, \Big[: \phi^3(x_1)\phi^3(x_2) :$$

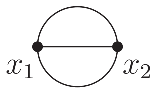


$$+ \; 3^2 : \phi^2(x_1)\phi^2(x_2) : \overbracket{\phi(x_1)\phi(x_2)} \quad + \quad 2(3)^2 : \phi(x_1)\phi(x_2) : \Big(\overbracket{\phi(x_1)\phi(x_2)}\Big)^2$$





$$+\ 3!\left(\contraction{}{\phi}{(x_1)}{\phi}\phi(x_1)\phi(x_2)\right)^3\Bigg]\quad+\quad\ldots \tag{6.44}$$



(6.45)

Feynman rules for graphical representation of the terms $\propto g^n$:

1. Draw all possible diagrams with n 3-point vertices, connected in all possible ways by lines (including disconnected diagrams).

2. Translate graphs into analytical expressions as follows:

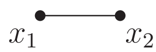
   - External lines for non-contracted fields:

$$\phi(x)\ =$$

(6.46)

   - Internal lines for contracted fields (=propagators):

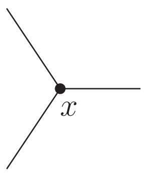
$$\contraction{}{\phi}{(x_1)}{\phi}\phi(x_1)\phi(x_2)\ =$$



(6.47)

   - Vertices for interaction terms:

$$\frac{\mathrm{i}g}{3!}\ =$$



(6.48)

3. Include a combinatorial factor (*symmetry factor*) for each diagram (more details explained below).

4. Integrate the sum of all terms according to

$$\frac{1}{n!}\int \mathrm{d}^4x_1\ldots\mathrm{d}^4x_n :\ \ldots\ : . \tag{6.49}$$

6.3.3 Feynman rules for S-matrix elements

Final task: Evaluate $\langle f|S|i\rangle$ upon sandwiching expansion of S-operator between states

$$|i\rangle = |\vec{k}_1,\ldots\vec{k}_m\rangle\ , \qquad\qquad |f\rangle = |\vec{p}_1,\ldots\vec{p}_n\rangle\,. \tag{6.50}$$

Definitions:

- *T-matrix*

$$\langle f|S|i\rangle \quad = \quad \underbrace{\langle f|i\rangle}_{\substack{=0 \text{ for } |i\rangle \neq |f\rangle\,, \\ \text{unscattered part}}} \quad + \quad \underbrace{\langle f|\, S - \mathbf{1}\,|i\rangle}_{\substack{\equiv \langle f|T|i\rangle,\ T\text{-matrix}, \\ \text{only scattered part}}} \quad . \tag{6.51}$$

$\hookrightarrow$ Only the T-matrix contributes to a non-trivial scattering with $|i\rangle \neq |f\rangle$.

- *Transition matrix element (transition amplitude)*

$$\langle f|T|i\rangle \quad = \quad \mathrm{i}(2\pi)^4 \underbrace{\delta\Big(\sum_i k_i - \sum_j p_j\Big)}_{\substack{\text{expresses momentum} \\ \text{conservation, due to} \\ \text{translational invariance}}} \quad \underbrace{\mathcal{M}_{fi}}_{\substack{\text{transition matrix} \\ \text{element}}} \quad . \tag{6.52}$$

Example: $2 \to 2$ **scattering**

$$|i\rangle = |\vec{k}_1, \vec{k}_2\rangle\ , \qquad\qquad |f\rangle = |\vec{p}_1, \vec{p}_2\rangle\,. \tag{6.53}$$

Use the expansion of the S matrix (6.44).
$\hookrightarrow$ $\langle f|T|i\rangle$ involves expectation values of normal-ordered operator products:

$$\langle \vec{p}_1, \vec{p}_2| : \phi^3(x) : |\vec{k}_1, \vec{k}_2\rangle\ , \quad \langle \vec{p}_1, \vec{p}_2| : \phi^n(x_1)\phi^n(x_2) : |\vec{k}_1, \vec{k}_2\rangle\,, \quad n = 0, 1, 2, 3. \tag{6.54}$$

Recall: $\quad \phi = \phi_{\text{in}} =$ free field.
$\hookrightarrow$ Use plane-wave decomposition with creation/annihilation operators $a^\dagger(\vec{p})/a(\vec{p})$:

$$\phi(x) = \int \mathrm{d}\tilde{p}\ \left[a(\vec{p})\, e^{-\mathrm{i}px} + a^\dagger(\vec{p})\, e^{\mathrm{i}px}\right]. \tag{6.55}$$

$\Rightarrow$ Only operator combinations with two a's and two $a^\dagger$'s contribute in $\langle \vec{p}_1, \vec{p}_2| \ldots |\vec{k}_1, \vec{k}_2\rangle$, i.e.

$$\langle \vec{p}_1, \vec{p}_2| : \phi^2(x_1)\phi^2(x_2) : |\vec{k}_1, \vec{k}_2\rangle\,. \tag{6.56}$$

Typical manipulation:

$$
\begin{aligned}
: \ldots \phi(x_i)\phi(x_j) : |\vec{k}_1, \vec{k}_2\rangle &= \int \mathrm{d}\tilde{q}_1\, \mathrm{d}\tilde{q}_2\, \mathrm{e}^{-\mathrm{i}q_1 x_i}\, \mathrm{e}^{-\mathrm{i}q_1 x_j}\, a(\vec{q}_1) a(\vec{q}_2) \underbrace{|\vec{k}_1, \vec{k}_2\rangle}_{=\ a^\dagger(\vec{k}_1)a^\dagger(\vec{k}_2)\,|0\rangle} \quad + \quad \ldots \\
&= \int \mathrm{d}^3q_1\, \mathrm{d}^3q_2\, \mathrm{e}^{-\mathrm{i}q_1 x_i}\, \mathrm{e}^{-\mathrm{i}q_1 x_j} \Big[\delta(\vec{q}_2 - \vec{k}_1)\, \delta(\vec{q}_1 - \vec{k}_2) + \delta(\vec{q}_2 - \vec{k}_2)\, \delta(\vec{q}_1 - \vec{k}_1)\Big]\, |0\rangle \ + \ \ldots \\
&= \Big(\mathrm{e}^{-\mathrm{i}k_1 x_i} \mathrm{e}^{-\mathrm{i}k_2 x_j} + \mathrm{e}^{-\mathrm{i}k_1 x_j} \mathrm{e}^{-\mathrm{i}k_2 x_i}\Big)\, |0\rangle \ + \ \ldots
\end{aligned} \tag{6.57}
$$

General case:

- Define contractions of fields with external states:

$$: \ldots \phi(x) \ldots : | \ldots \vec{k} \ldots \rangle \;=\; \mathrm{e}^{-\mathrm{i}kx}, \tag{6.58}$$

$$\langle \ldots \vec{p} \ldots | : \ldots \phi(x) \cdots : \;=\; \mathrm{e}^{\mathrm{i}px}. \tag{6.59}$$

- Perform all possible contractions of fields in normal-ordered products with external states.

Application to example $\langle \vec{p}_1, \vec{p}_2 | : \phi^2(x_1)\phi^2(x_2) : | \vec{k}_1, \vec{k}_2 \rangle$:

- Three types of contractions:

$$\langle \vec{p}_1, \vec{p}_2 | : \phi^2(x_1)\phi^2(x_2) : | \vec{k}_1, \vec{k}_2 \rangle \;=\; 2^2\, \mathrm{e}^{\mathrm{i}(p_1+p_2)x_1}\, \mathrm{e}^{-\mathrm{i}(k_1+k_2)x_2}, \tag{6.60}$$

$$\langle \vec{p}_1, \vec{p}_2 | : \phi^2(x_1)\phi^2(x_2) : | \vec{k}_1, \vec{k}_2 \rangle \;=\; 2^2\, \mathrm{e}^{-\mathrm{i}(k_1-p_1)x_1}\, \mathrm{e}^{-\mathrm{i}(k_2-p_2)x_2}, \tag{6.61}$$

$$\langle \vec{p}_1, \vec{p}_2 | : \phi^2(x_1)\phi^2(x_2) : | \vec{k}_1, \vec{k}_2 \rangle \;=\; 2^2\, \mathrm{e}^{-\mathrm{i}(k_1-p_2)x_1}\, \mathrm{e}^{-\mathrm{i}(k_2-p_1)x_2}, \tag{6.62}$$

  plus the identical contributions with $x_1 \leftrightarrow x_2$. $\Rightarrow$ Factor of 2.

- Apply remaining factors and integrals for T-matrix element:

$$\frac{1}{2} 3^2 \frac{(\mathrm{i}g)^2}{(3!)^2} \int \mathrm{d}^4x_1\, \mathrm{d}^4x_2\; \langle \vec{p}_1, \vec{p}_2 | : \phi^2(x_1)\phi^2(x_2) : | \vec{k}_1, \vec{k}_2 \rangle\; \phi(x_1)\phi(x_2) \tag{6.63}$$

  and use the momentum-space representation of propagator,

$$\phi(x_1)\phi(x_2) = \langle 0 | T\left[\phi(x_1)\phi(x_2)\right] | 0 \rangle = \int \frac{\mathrm{d}^4q}{(2\pi)^4} \frac{\mathrm{i}}{q^2 - m^2 + \mathrm{i}\epsilon} \mathrm{e}^{-\mathrm{i}q(x_1-x_2)}. \tag{6.64}$$

  Example: Explicit evaluation of contribution (6.62):

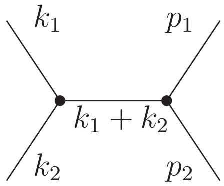
$$\begin{aligned}
&\frac{1}{2} 3^2 \frac{(\mathrm{i}g)^2}{(3!)^2} \int \mathrm{d}^4x_1\, \mathrm{d}^4x_2 \int \frac{\mathrm{d}^4q}{(2\pi)^4} \frac{\mathrm{i}}{q^2 - m^2 + \mathrm{i}\epsilon} \mathrm{e}^{-\mathrm{i}q(x_1-x_2)}\, 2^3\, \mathrm{e}^{-i(k_1-p_2)x_1}\, \mathrm{e}^{-i(k_2-p_1)x_2} \\
&\quad = (\mathrm{i}g)^2 \int \frac{\mathrm{d}^4q}{(2\pi)^4} (2\pi)^4 \delta(q + k_1 - p_2) \frac{i}{q^2 - m^2 + \mathrm{i}\epsilon} (2\pi)^4 \delta(q + p_1 - k_2) \\
&\quad = (2\pi)^4 \delta(k_1 + k_2 - p_1 - p_2)\, (\mathrm{i}g)^2 \frac{\mathrm{i}}{(k_1 - p_2)^2 - m^2 + \mathrm{i}\epsilon},
\end{aligned} \tag{6.65}$$

  $\hookrightarrow$ δ-function for momentum conservation appears explicitly [cf. (6.52)]; all combinatorial prefactors have canceled.

- Sum of the three different contractions (6.60)–(6.62):

$$\mathrm{i}\mathcal{M}_{fi} = (\mathrm{i}g)^2 \left[\frac{\mathrm{i}}{(k_1+k_2)^2 - m^2 + \mathrm{i}\epsilon} + \frac{\mathrm{i}}{(k_1-p_1)^2 - m^2 + \mathrm{i}\epsilon} + \frac{\mathrm{i}}{(k_1-p_2)^2 - m^2 + \mathrm{i}\epsilon} \right]$$

$=$



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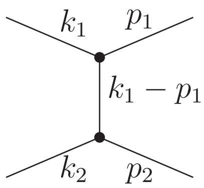

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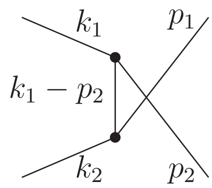


$$= (\mathrm{i}g)^2 \left[\frac{\mathrm{i}}{s - m^2 + \mathrm{i}\epsilon} + \frac{\mathrm{i}}{t - m^2 + \mathrm{i}\epsilon} + \frac{\mathrm{i}}{u - m^2 + \mathrm{i}\epsilon} \right], \tag{6.66}$$

with the three Mandelstam variables (see Exercise 2.3):

$$s = (k_1+k_2)^2\,, \qquad t = (k_1-p_1)^2\,, \qquad u = (k_1-p_2)^2\,. \tag{6.67}$$

Note: One diagram of the expansion (6.44) of the S-operator produces three diagrams in the expansion of the transition matrix element.

Generalization of the $2 \to 2$ example to arbitrary processes leads to

Feynman rules transition matrix elements in momentum space

for contributions proportional to g^N term in $\mathrm{i}\mathcal{M}_{fi}$ for an $n \to m$ scattering process:

1. Draw all possible diagrams with N 3-point vertices and n incoming and m outgoing external legs.

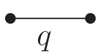
2. Impose momentum conservation at each vertex.

3. Insert the following expressions:

   - external lines:

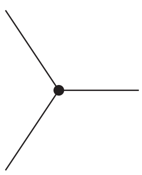
$$= 1 \tag{6.68}$$

   - internal lines:



$$= \frac{\mathrm{i}}{q^2 - m^2 + \mathrm{i}\epsilon} \tag{6.69}$$

   - vertices:



$$= \mathrm{i}g \tag{6.70}$$

4. Apply a symmetry factor $1/S_G$ for each diagram (see below).
5. Integrate over (loop) momenta q_i not fixed by momentum conservation according to

$$\int \frac{\mathrm{d}^4 q_i}{(2\pi)^4}\,. \tag{6.71}$$

Note: In our $2 \to 2$ example $S_G = 1$, and all momenta were fixed by external momenta.

Comments to the general case:

- *Loop momenta* (not fixed by external states):

  Systematic counting:

  - 1 momentum integral per propagator.
  - 1 space–time integral per vertex (yields momentum conservation at vertex).
    Above example: $\displaystyle\int \mathrm{d}^4x_1\, \mathrm{e}^{-\mathrm{i}(q+k_1-p_2)x_1} = (2\pi)^4\delta(q+k_1-p_2)$.
  - δ-function for overall momentum conservation split off from $\mathcal{M}_{fi}$.

  $\Rightarrow$ remaining # momentum integrals is given by

  $$L \;=\; \#\text{propagators} - \#\text{vertices} + 1 \;=\; \textit{number of loops in a diagram.} \tag{6.72}$$

  $\Rightarrow$ Perturbation series for $\mathcal{M}_{fi}$ is an expansion in # loops:

  - $L = 0$, leading order, *Born approximation*,
  - $L = 1$, next-to-leading order, *one-loop approximation*,

  ...

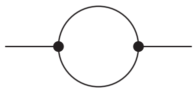
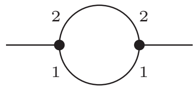
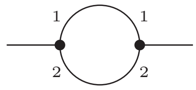
- *Symmetry factor S_G:*

  $S_G \neq 1$ results from two sources:

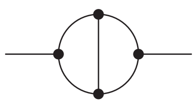
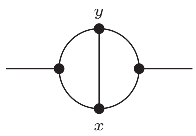
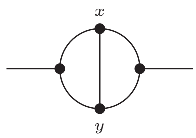
  - Incomplete cancellation of factor $1/3!$ in $\mathcal{H}_{\text{int}}(x) = -\dfrac{g}{3!} : \phi^3(x) :$,
    because some contractions to propagators are diagrammatically equivalent.
  - Incomplete cancellation of factor $1/n!$ in nth term of $S = T\exp\{...\}$,
    because some permutations of vertices x_i are diagrammatically equivalent.

  S_G = # permutations of internal lines or vertices that leaves the diagram unchanged
  = the order of the symmetry group of graph G.

Examples:

 $S_G = 2$, since  $\equiv$  etc.

$\hookrightarrow$ only $(3!)^2/2$ different contractions.

 $S_G = 2$, since  $\equiv$  etc.

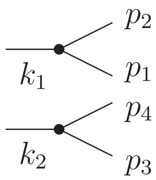
$\hookrightarrow$ only $4!/2$ different permutations of vertices.

- *Disconnected diagrams:*

  Momentum conservation at each vertex

  $\hookrightarrow$ "overall momentum conservation" in each connected part of a diagram.

  Example:

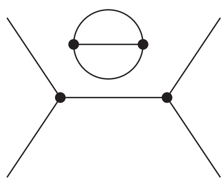
  $\propto \quad \delta(k_1 + p_1 + p_2)\, \delta(k_2 + p_3 + p_4)$.

  Contributions only for *exceptional momenta*, corresponding to different scattering proceeding in parallel.

  $\Rightarrow$ Contributions irrelevant for single scattering reaction.

- *Vacuum diagrams:*

  = (sub)diagrams with no external legs, e.g.:

  – Vacuum graphs always factor from remaining graphs of the diagram.

    $\hookrightarrow$ They modify each S-matrix element S_{fi} by the same factor.

  – Vacuum correction factor calculable as

$$\langle 0|S|0\rangle = \langle 0|U_I(\infty, -\infty)|0\rangle = \text{phase factor}, \qquad (6.73)$$

    contradicting the initial assumption (6.7) $|0\rangle = S\,|0\rangle$.

$\Rightarrow$ Redefine

$$S = \frac{U_I(\infty, -\infty)}{\langle 0|U_I(\infty, -\infty)|0\rangle},$$

so that $\langle 0|S|0\rangle = 1$ and all vacuum graphs cancel in observables (i.e. they can be ignored in practice).

6.4 Cross sections and decay widths

Cross section:

Definition:
$$\underbrace{\mathrm{d}N_s}_{\substack{\#\text{ scattering events / time} \\ \text{with particle } f_l \text{ carrying} \\ \text{momentum } \vec{p}_{f_l}}} = \underbrace{\mathrm{d}\sigma}_{\substack{\text{differential} \\ \text{cross section}}} \times \underbrace{F}_{\substack{\text{incoming particle flux} \\ = \frac{\#\text{interactions}}{\text{area}\times\text{time}}}},$$

where

- $F = \dfrac{N_1 N_2 v_{\text{rel}}}{V}$ with $N_{1,2} = \#$ incoming particles type $1, 2$ and $v_{\text{rel}} =$ relative velocity between incoming particles.

- $$\mathrm{d}N_s = \underbrace{\frac{W_{fi}}{T}}_{\text{transition rate}} \prod_{l=1}^{n} \left(V \frac{\mathrm{d}^3 p_{f_l}}{(2\pi)^3} \right) N_1 N_2 \,,$$

  with $V \dfrac{\mathrm{d}^3 p_{f_l}}{(2\pi)^3} = \#$ states of stationary waves in a box with volume V.

- Transition probability W_{fi} for $|i\rangle \to |f\rangle$:

$$W_{fi} = \frac{|\langle f|T|i\rangle|^2}{\langle f|f\rangle \langle i|i\rangle} \tag{6.74}$$

which is not well defined as $|\langle f|T|i\rangle|^2 \propto [\delta(p_i - p_f)]^2$ and $\langle f|f\rangle, \langle i|i\rangle \to \infty$.

Solution:
Box with finite extension in space–time, volume $= V \cdot T$:

$$\begin{aligned} (2\pi)^8 [\delta(p_i - p_f)]^2 &\longrightarrow V \cdot T \, (2\pi)^4 \delta(p_i - p_f), \\ \langle \vec{p}|\vec{p}'\rangle = (2\pi)^3 2p^0 \delta(\vec{p} - \vec{p}') &\xrightarrow{\vec{p}' \to \vec{p}} 2p^0 V. \end{aligned} \tag{6.75}$$

$$\Rightarrow \quad W_{fi} \;=\; V\cdot T\,(2\pi)^4\delta(p_i-p_f)\,\frac{1}{(2p^0_{i_1}V)(2p^0_{i_2}V)}\left(\prod_{l=1}^{n}\frac{1}{(2p^0_{f_l}V)}\right)\,|\underbrace{\mathcal{M}_{fi}}_{\substack{\text{transition}\\ \text{matrix element}}}|^2,$$

$$\mathrm{d}\sigma \;=\; \underbrace{\frac{1}{4p^0_{i_2}p^0_{i_2}v_{\mathrm{rel}}}}_{\text{flux factor}}\;|\mathcal{M}_{fi}|^2\;\underbrace{\left(\prod_{l=1}^{n}\frac{\mathrm{d}^3p_{f_l}}{2p^0_{f_l}(2\pi)^3}\right)(2\pi)^4\delta(p_i-p_f)}_{=\,\mathrm{d}\Phi_f,\ \text{invariant phase space volume}}\,. \tag{6.76}$$

Using the Lorentz-invariant form of F,

$p^0_{i_1}p^0_{i_2}v_{\mathrm{rel}} = \sqrt{(p_{i_1}p_{i_2})^2-(m_{i_1}m_{i_2})^2}$, with $p^2_{i_1}=m^2_{i_1}$, $p^2_{i_2}=m^2_{i_2}$,

$\mathrm{d}\sigma$ takes the final form:

$$\mathrm{d}\sigma \;=\; \frac{1}{4\sqrt{(p_{i_1}p_{i_2})^2-(m_{i_1}m_{i_2})^2}}\,|\mathcal{M}_{fi}|^2\,\mathrm{d}\Phi_f. \tag{6.77}$$

$\Rightarrow$ Total cross section:

$$\sigma_{\mathrm{tot}} = \int \mathrm{d}\sigma \;=\; \frac{1}{4\sqrt{(p_{i_1}p_{i_2})^2-(m_{i_1}m_{i_2})^2}}\int \mathrm{d}\Phi_f\,|\mathcal{M}_{fi}|^2. \tag{6.78}$$

Comments:

- $\mathrm{d}\sigma$ is Lorentz invariant.
- For polarizable particles ($\neq$ scalars):
  - initial state: take specific polarization or average over incoming spin states,
  - final state: analyze specific polarization or sum over outgoing spin states.
- Identical particles in final state: exclude identical configurations in $\int \mathrm{d}\Phi_f$.

  $\hookrightarrow$ Factor $1/(n_x!)$ for n_X identical particles of type X in full integral.
- Differential cross sections:
  Leave one or more kinematical variables in $\int \mathrm{d}\Phi_f$ open.

  Example: Distribution in scattering angle θ for the particle f_1:

  $$\frac{\mathrm{d}\sigma}{\mathrm{d}\theta} \;=\; \int \mathrm{d}\sigma\,\delta(\theta-\theta_{f_1}).$$

Decay width:

Particle decay: $X \to f_1 + \cdots + f_n$

$\hookrightarrow$ Treatment analogous to scattering!

Results:

- Partial decay width:

$$\Gamma_{X\to f} \;=\; \frac{1}{2m_X}\int \mathrm{d}\Phi_f\, |\mathcal{M}_{fX}|^2. \tag{6.79}$$

- Total decay width:

$$\Gamma_{\text{tot}} \;=\; \underbrace{\sum_f}_{\substack{\text{sum over all}\\ \text{decay channels}}} \Gamma_{X\to f}, \tag{6.80}$$

$\Rightarrow$ Particle lifetime: $\tau_X = \dfrac{\hbar}{\Gamma_{\text{tot}}}$.

Note: Treatment of polarizations, identical particles, and differential distributions analogous to cross section.

Example: $\phi\phi$ scattering in ϕ^3 theory in lowest order

Process:

$$\phi(k_1)\,\phi(k_2) \;\to\; \phi(p_1)\,\phi(p_2), \qquad \text{momentum conservation: } k_1+k_2 = p_1+p_2. \tag{6.81}$$

- Momenta in centre-of-mass frame:

$$k_{1,2}^\mu = (E,0,0,\pm\beta E), \quad k_{1,2}^2 = m^2, \tag{6.82}$$

with $E =$ beam energy, $\beta = \sqrt{1-\dfrac{m^2}{E^2}} =$ velocity,

$$p_{1,2}^\mu = (E, \pm E\beta\sin\theta\cos\varphi, \pm E\beta\sin\theta\sin\varphi, \pm E\beta\cos\theta), \quad p_{1,2}^2 = m^2, \tag{6.83}$$

with $\theta =$ *scattering angle* $=$ angle between $\vec{p}_1$ and $\vec{k}_1$.

- Mandelstam variables:

$$s = (k_1+k_2)^2 = 4E^2, \tag{6.84}$$

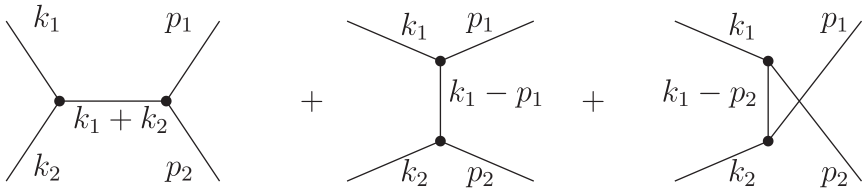
$$t = (k_1-p_1)^2 = p_1^2 + k_1^2 - 2p_1\cdot k_1 = -2\beta^2E^2(1-\cos\theta) = -4\beta^2E^2\sin^2\frac{\theta}{2}, \tag{6.85}$$

$$u = (k_1-p_2)^2 = \cdots = -4\beta^2E^2\cos^2\frac{\theta}{2}, \tag{6.86}$$

$$s+t+u = 4m^2. \tag{6.87}$$

- Born diagrams:

$$\mathrm{i}\mathcal{M}_{fi} = \tag{6.88}$$



s-channel t-channel u-channel

$$= (\mathrm{i}g)^2 \left[\frac{\mathrm{i}}{s-m^2} + \frac{\mathrm{i}}{t-m^2} + \frac{\mathrm{i}}{u-m^2} \right] \tag{6.89}$$

= dependent on E and θ, but not on φ (rotational invariance wrt beam axis!).

- Cross section:
  - flux $= \dfrac{1}{4\sqrt{(k_1 k_2)^2 - m^4}} = \dfrac{1}{4\sqrt{E^4(1+\beta^2)^2 - m^4}} = \dfrac{1}{8E^2\beta}$,
  - phase space: (see Exercise 5.2)

$$\int \mathrm{d}\Phi_2 = \frac{1}{(2\pi)^2} \frac{\sqrt{\lambda(s, m^2, m^2)}}{8s} \int \mathrm{d}\Omega_1 = \frac{\beta}{32\pi^2} \int \mathrm{d}\varphi \int \mathrm{d}\cos\theta. \tag{6.90}$$

$$\begin{aligned} \Rightarrow \quad \sigma &= \text{flux} \int \mathrm{d}\Phi_2 \, |\mathcal{M}_{fi}|^2 \\ &= \frac{g^4}{256\pi^2 E^2} \underbrace{\int \mathrm{d}\varphi}_{\to 2\pi} \underbrace{\int \mathrm{d}\cos\theta \left[\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2} \right]^2}_{\text{expressible in terms of logarithms}}. \end{aligned} \tag{6.91}$$

Part II

Quantization of fermion fields

Chapter 7

Representations of the Lorentz group

7.1 Lie groups and algebras

$\hookrightarrow$ Continuous groups (e.g. Lorentz / Poincaré groups, many internal symmetry groups)

7.1.1 Definitions

A *Lie group* is a group whose elements g are parametrized by a set of continuous parameters ω_a , $a = 1, \dots n$: $g(\omega) = g(\omega_1, \dots, \omega_n)$.

- Group-multiplication law:

$$g(\omega)g(\omega') = g(\omega'') \quad \text{with} \quad \omega''_a = \underbrace{f_a(\omega, \omega')}_{\text{differentiable functions of all } \omega_b, \omega'_c}, \tag{7.1}$$

- n = dimension of the Lie group,
- identity e corresponds to $\omega_a = 0$ by convention:

$$g(0) = e. \tag{7.2}$$

- A Lie group is called *compact* if the set of all ω_a is compact.
- A Lie group is called *connected* if every element $g(\omega)$ is connected to the identity e by a continuous path in the set of the parameters ω.

Example: Lorentz group L

- L = non-compact (space of boosts is non-compact).
- $L \neq$ connected (disconnected parts characterized by $\det \Lambda = \pm\mathbf{1}$ and $\Lambda^0{}_0 \gtrless 0$); $L^\uparrow_+$ = connected.

Comment:
Examples:

$GL(N, \mathbb{C})$**:** *general linear group*
The $2N^2$-dimensional group of invertible complex matrices A ($\det A \neq 0$).

$SL(N, \mathbb{C})$**:** *special linear group*
The $(2N^2 - 2)$-dimensional group of complex matrices A with $\det A = 1$.

$O(N)$**:** *orthogonal group*
The $N(N-1)/2$-dimensional group of real orthogonal matrices M ($M^T M = \mathbf{1}$, so that $\det M = \pm 1$).

$SO(N)$**:** *special orthogonal group*
The $N(N-1)/2$-dimensional group of orthogonal matrices M with $\det M = 1$.

$U(N)$**:** *unitary group*
The N^2-dimensional group of unitary matrices U, $U^\dagger U = 1$.

$SU(N)$**:** *special unitary group*
The $(N^2 - 1)$-dimensional group of unitary matrices U with $\det(U) = 1$.

Definition:
A *representation* D of a group G on a vector space V is a mapping of all $g \in G$ to linear transformations $D(g)$ on V that is compatible with group multiplication:

$$f \cdot g = h \quad \Rightarrow \quad D(f)D(g) = D(f \cdot g) = D(h). \tag{7.3}$$

- $V =$ *representation space*.
- $\dim V =$ dimension of the representation (if $n = \dim V < \infty$, $D(g)$ are $n \times n$-matrices).
- Elements $v \in V$ are called *multiplets* (at least in physics).
- Two representations D and D' are called *equivalent* ($D \sim D'$) if there is an invertible transformation S so that
$$D'(g) = SD(g)S^{-1}, \quad \forall g \in G.$$
- A representation is called *unitary* if the matrices $D(g)$ are unitary for all g.

7.1.2 Lie algebras

Note: Neighbourhood of identity carries almost full information about Lie group G.

Definition:
Lie algebra $\mathfrak{g} \equiv$ set of infinitesimal deviations from identity e
$\hookrightarrow$ vector space with product structure.

Properties of $\mathfrak{g}$ in a specific representation $D(\mathfrak{g})$:

- Infinitesimal group elements $g = g(\delta\omega)$ in representation $D(G)$:

$$D(g) \equiv D(\delta\omega) = \mathbf{1} + \delta\omega_a \underbrace{\frac{\partial D(\omega)}{\partial \omega_a}\bigg|_{\omega=0}}_{\equiv -\mathrm{i}T_D^a} + \mathcal{O}(\delta\omega^2) \;=\; \mathbf{1} - \mathrm{i}T_D^a \delta\omega_a \;+\; \mathcal{O}(\delta\omega^2). \tag{7.4}$$

  $\{T_D^a\}$ = *generators* of the Lie group in D representation
  = basis for representation $D(\mathfrak{g})$ of $\mathfrak{g}$.

- Finite transformations (connected to the unit element) via exponentiation:

$$D(\omega) = \lim_{n\to\infty} \left(1 - \mathrm{i}\frac{\omega_a}{n} T^a\right)^n = \exp\left(-\mathrm{i}\omega_a T^a\right), \tag{7.5}$$

- Composition law of G implies product in $\mathfrak{g}$:

  Ansatz for composition functions of Eq. (7.1):

$$f_a(\omega, \omega') \;=\; \omega_a + \omega'_a + \frac{1}{2} f^{abc}\, \omega_b\, \omega'_c \;+\; \ldots, \tag{7.6}$$
$$\text{i.e. } f_a(\omega_b, 0) = f_a(0, \omega_b) = \omega_a.$$

  $\hookrightarrow$ Insertion into composition law:

$$\begin{aligned}
D(\omega)D(\omega') &= \exp\left(-\mathrm{i}\omega_a T^a\right) \exp\left(-\mathrm{i}\omega'_b T^b\right) \\
&= \mathbf{1} - \mathrm{i}\omega_a T^a - \mathrm{i}\omega'_b T^b - \frac{1}{2}(\omega_a T^a)^2 - \frac{1}{2}(\omega'_b T^b)^2 - \omega_a \omega'_b T^a T^b + \ldots \\
&\stackrel{!}{=} \exp\left(-\mathrm{i} f_a(\omega, \omega') T^a\right) \\
&= \mathbf{1} - \mathrm{i}\left(\omega_a + \omega'_a + \frac{1}{2} f^{abc}\, \omega_b\, \omega'_c\right) T^a - \frac{1}{2}(\omega_a + \omega'_a)(\omega_b + \omega'_b) T^a T^b + \ldots
\end{aligned}$$

$$\text{i.e.} - \omega_a \omega'_b T^a T^b \;=\; -\frac{\mathrm{i}}{2} f^{abc}\, \omega_b\, \omega'_c T^a - \frac{1}{2}\omega_a \omega'_b T^a T^b - \frac{1}{2}\omega'_a \omega_b T^a T^b$$

  $\Rightarrow$ Basic Lie algebra relation:

$$T^b T^c - T^c T^b \;\equiv\; [T^b, T^c] = \mathrm{i}\, \underbrace{f^{abc}}_{\textit{structure constants} \text{ of } \mathfrak{g} \,=\, \text{independent of representation !}}\, T^a, \quad \text{where } f^{abc} = -f^{acb}. \tag{7.7}$$

- *Jacobi identity* of commutators,

$$[T^a,[T^b,T^c]]+[T^c,[T^a,T^b]]+[T^b,[T^c,T^a]]=0, \tag{7.8}$$

  implies

$$f^{abk}f^{kcd}+f^{ack}f^{kdb}+f^{adk}f^{kbc}=0. \tag{7.9}$$

- *Adjoint representation*

$$(T^a_{\text{adj}})_{bc}\equiv -\mathrm{i}f^{bca} \tag{7.10}$$

  exists for each Lie algebra.
  Commutator relation (7.7) satisfied due to the Jacobi identiy (7.9).

Important special case: algebra of a *compact* (and semispimple) Lie group

- Matrix $\mathrm{tr}(T^aT^b)$ is positive definite.

  $\hookrightarrow$ Convention:

$$\mathrm{tr}(T^aT^b)=T_D\delta_{ab}, \qquad T_D = \textit{Dynkin index } = \tfrac{1}{2} \text{ in defining representation.} \tag{7.11}$$

  $\Rightarrow$ f^{abc} are totally antisymmetric, since

$$f^{abc}=-2\mathrm{i}\,\mathrm{tr}\left((T^bT^c-T^cT^b)T^a\right)=-2\mathrm{i}\,\mathrm{tr}\left(T^bT^cT^a-T^cT^bT^a\right)=\text{cyclic in } abc.$$

- Finite-dimensional representations are unitary:

$$D(\omega)^\dagger=\exp\left(\mathrm{i}\omega^aT^{a\dagger}\right)\overset{!}{=}D(\omega)^{-1}=\exp\left(\mathrm{i}\omega^aT^a\right), \quad \text{i.e. } T^a=T^{a\dagger}. \tag{7.12}$$

  $\Rightarrow$ Generators T^a are hermitian.

> Comment:
> Defining representations of matrix Lie algebras:
> $T^a = N\times N$ matrices with special properties:
>
> - $GL(N,\mathbb{C})$: complex, no restriction
> - $SL(N,\mathbb{C})$: complex, traceless
> - $SO(N)$: imaginary, antisymmetric
> - $SU(N)$: hermitian, traceless

Example: $SU(2)$ = relevant group for angular momentum in QM

- Group elements in the defining 2-dim. (*fundamental*) representation:
  U = unitary 2×2 matrix with $\det U = 1$.
- Generators in the *fundamental representation*:
  T^a = traceless, hermitian 2×2 matrices ($a = 1, 2, 3$).
  Usual convention: $T^a = \frac{\sigma^a}{2}$, σ^a = Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.13}$$

  Lie algebra:

$$[\sigma^a, \sigma^b] = 2\mathrm{i}\epsilon^{abc}\sigma^c, \quad \text{structure constants} = \epsilon^{abc} = \text{totally antisym. } \epsilon\text{-tensor} \tag{7.14}$$

- Finite group elements in fundamental representation:

$$U(\vec{\omega}) \;=\; \exp\left(-\frac{\mathrm{i}}{2}\vec{\omega}\cdot\vec{\sigma}\right) = \cos\left(\frac{\omega}{2}\right) - \mathrm{i}\vec{\mathrm{e}}\cdot\vec{\sigma}\sin\left(\frac{\omega}{2}\right), \tag{7.15}$$

  where $\vec{\omega} = \omega\vec{\mathrm{e}}$ ($\vec{\mathrm{e}}^{\,2} = 1$).

- Adjoint representation:

$$T^1_{\mathrm{adj}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad T^2_{\mathrm{adj}} = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}, \quad T^3_{\mathrm{adj}} = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7.16}$$

  Finite group elements:
  $R(\vec{\omega}) = \exp(-\mathrm{i}T^a_{\mathrm{adj}}\omega^a)$ = 3dim. rotation matrices with angle ω around axis $\vec{\mathrm{e}}$.

- Any representation: $T^a = J^a$ = components of angular momentum.

7.1.3 Irreducible representations

Definitions:

- A representation is called *reducible* if there is a subspace H of V that is invariant under all matrices $D(g)$, i.e. if all $D(g)$ can be brought (via some similiarity transformation) into the block form

$$D(g) = \begin{pmatrix} D_1(g) & E(g) \\ 0 & D_2(g) \end{pmatrix}. \tag{7.17}$$

- If there is no invariant subspace, the representation is called *irreducible*.

- A representation is called *fully reducible* if all $D(g)$ can be written in block-diagonal form,
$$D(g) = \begin{pmatrix} D_1(g) & 0 & 0 & \dots \\ 0 & D_2(g) & 0 & \dots \\ & & \ddots & \\ 0 & \dots & & D_n(g) \end{pmatrix}, \tag{7.18}$$
where the D_n are irreducible, i.e. a fully reducible representation is the direct sum of irreducible representations:
$$D = D_1 \oplus D_2 \oplus \cdots \oplus D_n. \tag{7.19}$$
- Definitions of (ir)reducibility for Lie algebras analogously.
- An operator C commuting with all elements of the Lie algebra is called *Casimir operator*:
$$[C, T^a] = 0. \tag{7.20}$$

Some facts about (ir)reducibility:

- Irreducible representations of abelian groups are one-dimensional.
- All unitary reducible group representations are fully reducible.
- *Schur's Lemma*:
If a linear mapping A on a vector space V commutes with all matrices $D(g)$ of an irreducible representation of the group G on V, i.e.
$$AD(g) = D(g)A \tag{7.21}$$
for all $g \in G$, then A is a multiple of the identity:
$$A = \lambda_D \mathbf{1}, \tag{7.22}$$
where λ_D depends on the representation.
- Schur's Lemma applied to Lie algebras:
Casimir operators $C \propto \mathbf{1}$ in an irreducible representation.
- In Lie algebras with f^{abc} = totally antisymmetric (e.g. for compact Lie groups) there is always the quadratic Casimir operator,
$$C_2 = T^a T^a. \tag{7.23}$$

Comment:
Proof that C_2 satisfies (7.20):
$$[C_2, T^b] = T^a[T^a, T^b] + [T^a, T^b]T^a \quad = \mathrm{i} f^{abc} \left(T^a T^c + T^c T^a\right) = 0. \tag{7.24}$$

Example: irreducible representations of $SU(2)$
$\hookrightarrow$ known from the angular momentum in QM

- Quadratic Casimir operator:

$$\vec{J}^2 = \sum_a J^a J^a = \text{total angular momentum operator}, \quad [\vec{J}^2, J^a] = 0. \tag{7.25}$$

$\Rightarrow$ Diagonalization of $\vec{J}^2$ and one component J^a possible, usual choice J^3.

- Irreducible representation $D^{(j)}$ for each fixed value of $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$:

$$\begin{aligned} \vec{J}^2 \, |j,m\rangle &= j(j+1)\, |j,m\rangle \,, \\ J^3 \, |j,m\rangle &= m\, |j,m\rangle \,, \qquad m = -j, -j+1, \ldots j. \end{aligned} \tag{7.26}$$

$\{|j,m\rangle\}$ = $(2j+1)$-dimensional multiplet for each fixed j.

7.1.4 Constructing representations

New group representations from two representations D_i $(i = 1,2)$ on vector spaces V_i $(\mathrm{dim}V_i = n_i)$:

- $D_1 \oplus D_2$ on the *direct sum* $V_1 \oplus V_2$ of vector spaces $(\mathrm{dim} = n_1 + n_2)$:

$$(D_1 \oplus D_2)(g) = \begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix}, \quad v_1 \oplus v_2 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad v_i \in V_i,$$
$$\text{i.e. } (D_1 \oplus D_2)(g)(v_1 \oplus v_2) = (D_1(g)v_1) \oplus (D_2(g)v_2). \tag{7.27}$$

Representation is reducible by construction.

- $D_1 \otimes D_2$ on the *tensor product* $V_1 \otimes V_2$ of vector spaces $(\mathrm{dim} = n_1 n_2)$:

$$(D_1 \otimes D_2)(g)(v_1 \otimes v_2) = (D_1(g)v_1) \otimes (D_2(g)v_2). \tag{7.28}$$

Representation is in general reducible, but decomposible into irreducible blocks $D^{(i)}$:

$$D_1 \otimes D_2 = D^{(i_1)} \oplus \cdots \oplus D^{(i_n)}. \tag{7.29}$$

Definitions carry over to Lie algebras: $D(g) = \mathbf{1} - \mathrm{i}\omega_a T^a_D + \ldots$

- Direct sum representation on $V_1 \oplus V_2$:

$$T^a_{D_1 \oplus D_2} = \begin{pmatrix} T^a_{D_1} & 0 \\ 0 & T^a_{D_2} \end{pmatrix}, \quad T^a_{D_1 \oplus D_2}(v_1 \oplus v_2) = (T^a_{D_1} v_1) \oplus (T^a_{D_2} v_2). \tag{7.30}$$

- Tensor product representation on $V_1 \otimes V_2$:

$$T^a_{D_1 \otimes D_2}(v_1 \otimes v_2) = (T^a_{D_1} v_1) \otimes v_2 + v_1 \otimes (T^a_{D_2} v_2). \tag{7.31}$$

Example: product representations of $SU(2)$

$\hookrightarrow$ addition of two angular momenta $\vec{J}_i$ $(i = 1, 2)$ with respective multiplets $|j_i, m_i\rangle$:

$$\vec{J}\,|j_1, m_1\rangle \otimes |j_2, m_2\rangle = (\vec{J}_1\,|j_1, m_1\rangle) \otimes |j_2, m_2\rangle + |j_1, m_1\rangle \otimes (\vec{J}_2\,|j_2, m_2\rangle). \tag{7.32}$$

Decomposition into irreducible blocks: (*Clebsch–Gordan series*)

$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle = \sum_j c_{j,m}\, |j, m = m_1 + m_2\rangle \quad \text{with} \quad |j_1 - j_2| \le j \le j_1 + j_2, \tag{7.33}$$

in terms of representation spaces:

$$D^{(j_1)} \otimes D^{(j_2)} = D^{(|j_1-j_2|)} \oplus D^{(|j_1-j_2|+1)} \oplus \cdots \oplus D^{(j_1+j_2)}. \tag{7.34}$$

Specifically:

$$\begin{aligned} D^{(\frac{1}{2})} \otimes D^{(\frac{1}{2})} &= D^{(0)} \oplus D^{(1)}\,, \\ D^{(1)} \otimes D^{(1)} &= D^{(0)} \oplus D^{(1)} \oplus D^{(2)}, \quad \text{etc.} \end{aligned} \tag{7.35}$$

7.2 Irreducible representations of the Lorentz group

Recall: Lorentz transformations and generators (see Chap. 2)

$$\Lambda = \exp\Big\{-\mathrm{i}(\underbrace{\nu^k K^k}_{\text{boost}} + \underbrace{\varphi^k J^k}_{\text{rotation}})\Big\}. \tag{7.36}$$

Lie algebra of generators:

$$[J^i, J^j] = \mathrm{i}\epsilon^{ijk} J^k, \tag{7.37}$$

$$[J^i, K^j] = \mathrm{i}\epsilon^{ijk} K^k, \tag{7.38}$$

$$[K^i, K^j] = -\mathrm{i}\epsilon^{ijk} J^k. \tag{7.39}$$

Simplification by change of basis:

$$T^k_{1,2} = \frac{1}{2}(J^k \mp \mathrm{i}K^k). \qquad \Rightarrow \qquad [T^i_a, T^j_b] = \mathrm{i}\epsilon^{ijk}\, T^k_a\, \delta_{ab}. \tag{7.40}$$

$\Rightarrow$ Lie algebras of $L^\uparrow_+$ and $SU(2) \times SU(2)$ closely related (complex versions are identical).

Construction of irreducible representations of $L^\uparrow_+$: (analogy to $SU(2)$ case)

Two commuting generators: T^3_1, T^3_2; two Casimir operators: $\vec{T}^2_1$, $\vec{T}^2_2$.

$\hookrightarrow$ Multiplets $|j_1, m_1; j_2, m_2\rangle \equiv |j_1, m_1\rangle_1 \otimes |j_2, m_2\rangle_2$ span $(2j_1 + 1)(2j_2 + 1)$-dimensional irreducible representation $D^{(j_1,j_2)}$ for fixed j_1, j_2:

$$\begin{aligned} T^3_a\,|j_1, m_1; j_2, m_2\rangle &= m_a\,|j_1, m_1; j_2, m_2\rangle\,, \quad m_a = -j_a, -j_a + 1, \ldots j_a, \\ \vec{T}^2_a\,|j_1, m_1; j_2, m_2\rangle &= j_a(j_a + 1)\,|j_1, m_1; j_2, m_2\rangle\,, \quad j_a = 0, \tfrac{1}{2}, 1, \ldots . \end{aligned} \tag{7.41}$$

Lorentz transformations in $D^{(j_1,j_2)}$:

$$\Lambda^{(j_1,j_2)} = \exp\left(-\mathrm{i}(\vec{\varphi}+\mathrm{i}\vec{\nu})\vec{T}_1^{(j_1)}\right)\exp\left(-\mathrm{i}(\vec{\varphi}-\mathrm{i}\vec{\nu})\vec{T}_2^{(j_2)}\right), \qquad [T_1^k, T_2^l] = 0. \tag{7.42}$$

Comments:

- Hermitian $SU(2)$ generators $T_a^{(j_a)}$ constructed as in non-relativistic QM.
- Angular momentum $\vec{J} = \vec{T}_1 + \vec{T}_2$.
  $\hookrightarrow$ $D^{(j_1,j_2)}$ contains angular momenta $j = |j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2$.
- $\Lambda^{(j_1,j_2)}$ is unitary only for pure rotations ($\vec{\nu} = 0$).

Parity transformation:

Behaviour of generators: $P\colon\ \vec{J} \to \vec{J}$ (pseudo-vector),

$P\colon\ \vec{K} \to -\vec{K}$ (vector).

$\Rightarrow$ P interchanges the two $SU(2)$ factors:

$$\vec{T}_1 \underset{P}{\longleftrightarrow} \vec{T}_2, \qquad \text{i.e.} \quad P\colon\ D^{(j_1,j_2)} \to D^{(j_2,j_1)}. \tag{7.43}$$

$\Rightarrow$ P-invariant representations: $D^{(j,j)}$ and $D^{(j_1,j_2)} \oplus D^{(j_2,j_1)}$ for $j_1 \neq j_2$.

7.3 Fundamental spinor representations

- $D^{(\frac{1}{2},0)}$: *Right-chiral fundamental representation*

Generators:
$$T_1^{(\frac{1}{2}),i} = \frac{\sigma^i}{2}, \qquad T_2^{0,i} = 0. \tag{7.44}$$

Transformations:
$$\Lambda^{(\frac{1}{2},0)} = \exp\left(-\frac{\mathrm{i}}{2}(\vec{\varphi}+\mathrm{i}\vec{\nu})\vec{\sigma}\right) \equiv \Lambda_R. \tag{7.45}$$

Multiplets:
$$\chi = \begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}, \quad \textit{right-chiral Weyl spinors.} \tag{7.46}$$

- $D^{(0,\frac{1}{2})}$: *Left-chiral fundamental representation*

Generators:
$$T_1^{0,i} = 0, \qquad T_2^{(\frac{1}{2}),i} = \frac{\sigma^i}{2}. \tag{7.47}$$

Transformations:
$$\Lambda^{(0,\frac{1}{2})} = \exp\left(-\frac{\mathrm{i}}{2}(\vec{\varphi}-\mathrm{i}\vec{\nu})\vec{\sigma}\right) \equiv \Lambda_L. \tag{7.48}$$

Multiplets:
$$\bar{\chi} = \begin{pmatrix}\bar{\chi}_1\\ \bar{\chi}_2\end{pmatrix}, \quad \textit{left-chiral Weyl spinors.} \tag{7.49}$$

Properties of $\Lambda_{R,L}$:

- $\Lambda_{R,L}$ = complex 2×2 matrices with $\det\Lambda_{R,L}=1$, i.e. $\Lambda_{R,L}\in SL(2,\mathbb{C})$.
- Useful identities:

$$\Lambda_R^\dagger=\Lambda_L^{-1},\qquad \Lambda_L^\dagger=\Lambda_R^{-1}. \tag{7.50}$$

- Relation by complex conjugation:

$$\begin{aligned}\epsilon^{-1}\sigma^i\epsilon &= -\sigma^{i*},\qquad \epsilon=\mathrm{i}\sigma^2=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.\\ \Rightarrow\ \Lambda_R^* &= \exp\left(\frac{\mathrm{i}}{2}(\vec\varphi-\mathrm{i}\vec\nu)\vec\sigma^*\right)=\exp\left(-\frac{\mathrm{i}}{2}\epsilon^{-1}(\vec\varphi-\mathrm{i}\vec\nu)\vec\sigma\epsilon\right)=\epsilon^{-1}\Lambda_L\epsilon.\end{aligned} \tag{7.51}$$

$\Rightarrow$ Equivalence: $\quad \left(D^{(\frac12,0)}\right)^*\sim D^{(0,\frac12)},\quad$ i.e. $\quad D^{(\frac12,0)}\xleftrightarrow[\text{conjugation}]{\text{complex}}D^{(0,\frac12)}$.

> Comment:
> Construction of left (right) spinors from a right (left) spinors:
>
> $$\begin{aligned}\chi\in D^{(\frac12,0)}: &\quad (\epsilon\chi^*)\to\epsilon\Lambda_R^*\chi^*=\Lambda_L(\epsilon\chi^*),\quad \text{i.e.}\quad \epsilon\chi^*\in D^{(0,\frac12)},\\ \bar\chi\in D^{(0,\frac12)}: &\quad (\epsilon^{-1}\bar\chi^*)\to\epsilon^{-1}\Lambda_L^*\bar\chi^*=\Lambda_R(\epsilon^{-1}\bar\chi^*),\quad \text{i.e.}\quad (\epsilon^{-1}\bar\chi^*)\in D^{(\frac12,0)}.\end{aligned} \tag{7.52}$$

Dirac spinors

Parity-invariant representation for spin-$\frac12$ fermions: (e.g. needed for electromagnetism)

- *Dirac representation:*

$$D^{(\frac12,0)}\oplus D^{(0,\frac12)}\qquad \text{(irreducible under } L_+^\uparrow\otimes P) \tag{7.53}$$

- *Dirac spinor* ψ:

$$\psi=\begin{pmatrix}\chi\\ \bar\xi\end{pmatrix}. \tag{7.54}$$

  Lorentz transformation of ψ:

$$\psi\ \underset{\Lambda}{\to}\ \psi'=\begin{pmatrix}\Lambda_R\chi\\ \Lambda_L\bar\xi\end{pmatrix}=\underbrace{\begin{pmatrix}\Lambda_R&0\\0&\Lambda_L\end{pmatrix}}_{\equiv S(\Lambda)}\psi=S(\Lambda)\psi. \tag{7.55}$$

7.4 Product representations

Decomposition of products $D^{(j_1,j_2)} \otimes D^{(j_1',j_2')}$ into irreducible blocks:

$\hookrightarrow$ Possible upon using relations from $SU(2)$:

- From Clebsch–Gordan series of $SU(2)$:

$$\begin{aligned} &D^{(j,0)} \otimes D^{(j',0)} = D^{(j+j',0)} \oplus D^{(j+j'-1,0)} \oplus \cdots \oplus D^{(|j-j'|,0)}, \\ &D^{(0,j)} \otimes D^{(0,j')} \text{ analogously.} \end{aligned} \tag{7.56}$$

- Independence of $SU(2)$ factors, i.e. $[T_1^k, T_2^l] = 0$:

$$D^{(j_1,j_2)} = \underbrace{D^{(j_1,0)} \otimes D^{(0,j_2)}}_{\text{independent factors}}. \tag{7.57}$$

$$\begin{aligned} \Rightarrow\ D^{(j_1,j_2)} \otimes D^{(j_1',j_2')} &= D^{(j_1,0)} \otimes D^{(0,j_2)} \otimes D^{(j_1',0)} \otimes D^{(0,j_2')} \qquad (7.58) \\ &= \underbrace{D^{(j_1,0)} \otimes D^{(j_1',0)} \otimes D^{(0,j_2)} \otimes D^{(0,j_2')}}_{\text{use } SU(2) \text{ relation (7.56)}} \\ = \ \ldots \ = D^{(j_1+j_1',j_2+j_2')} &\oplus D^{(j_1+j_1'-1,j_2+j_2')} \oplus D^{(j_1+j_1',j_2+j_2'-1)} \cdots \oplus D^{(|j_1-j_1'|,|j_2-j_2'|)}. \end{aligned}$$

Note: Reduction important in construction of covariant quantities from products of multiplets (e.g. invariants for Lagrangians).

Examples:

- *Invariant spinor product:*

$$D^{(\frac{1}{2},0)} \otimes D^{(\frac{1}{2},0)} = D^{(1,0)} \oplus \underbrace{D^{(0,0)}}_{\substack{\text{trivial representation} \\ \text{(objects Lorentz invariant)}}} \tag{7.59}$$

$\Rightarrow$ $\exists$ 2×2 matrix $A = (a_{ij})$ so that

$$a_{ij}\chi_i'\xi_j' = a_{ij}(\Lambda_R)_{ik}(\Lambda_R)_{jl}\chi_k\xi_l \stackrel{!}{=} a_{ij}\chi_i\xi_j = \text{invariant}, \quad \text{i.e.} \quad \Lambda_R^T A \Lambda_R = A \quad \forall \Lambda_R \in L. \tag{7.60}$$

Solution: $A = \epsilon =$ totally antisymmetric tensor.

$\Rightarrow$ Lorentz-invariant product of Weyl spinors: (left-handed case analogously)

$$\langle \chi\xi \rangle \equiv \epsilon_{ij}\chi_i\xi_j, \qquad \langle \bar{\chi}\bar{\xi} \rangle \equiv \epsilon_{ij}\bar{\chi}_i\bar{\xi}_j. \tag{7.61}$$

- *4-vector representation:*

  Required: real, P-invariant representation that contains spin value $j = 1$ (vector!).

  Simplest candidate:

$$D^{(\frac{1}{2},\frac{1}{2})} = D^{(\frac{1}{2},0)} \otimes D^{(0,\frac{1}{2})} \sim D^{(\frac{1}{2},0)} \otimes \left(D^{(\frac{1}{2},0)}\right)^* \tag{7.62}$$

  To show: $\exists$ 2×2 matrices C^μ so that

$$\chi_i^\dagger C^\mu_{ij} \xi_j = \text{4-vector}, \qquad \text{i.e.} \quad \Lambda_R^\dagger C^\mu \Lambda_R = \Lambda^\mu{}_\nu C^\nu. \tag{7.63}$$

  Solution: (see Exercise 8.3)

$$C^\mu = \sigma^\mu = (\mathbf{1}, \sigma^1, \sigma^2, \sigma^3). \tag{7.64}$$

  Analogously:

$$\Lambda_L^\dagger \bar{\sigma}^\mu \Lambda_L = \Lambda^\mu{}_\nu \bar{\sigma}^\nu, \qquad \bar{\sigma}^\mu = (\mathbf{1}, -\sigma^1, -\sigma^2, -\sigma^3). \tag{7.65}$$

  $\Rightarrow$ 4-vectors from products of Weyl spinors:

$$\chi^\dagger \sigma^\mu \xi, \qquad \bar{\chi}^\dagger \bar{\sigma}^\mu \bar{\xi}. \tag{7.66}$$

- *Dirac representation:*

  Covariants from products of two Dirac spinors ψ_1, ψ_2 ?

$$\begin{aligned} &\left[D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}\right] \otimes \left[D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}\right] \\ &= \underbrace{D^{(0,0)}}_{\text{scalar}} \oplus \underbrace{D^{(0,0)}}_{\text{psudo-scalar}} \oplus \underbrace{D^{(\frac{1}{2},\frac{1}{2})}}_{\text{vector}} \oplus \underbrace{D^{(\frac{1}{2},\frac{1}{2})}}_{\text{pseudo-vector}} \oplus \underbrace{D^{(1,0)} \oplus D^{(0,1)}}_{\text{anti-sym. rank-2 tensors}}. \end{aligned} \tag{7.67}$$

  Auxiliary quantities for explicit construction:

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix} \qquad \textit{Dirac matrices} \text{ in } \textit{chiral representation}, \tag{7.68}$$

$$\overline{\psi} \equiv \psi^\dagger \gamma_0 = (\bar{\phi}^\dagger, \chi^\dagger) \qquad \textit{adjoint Dirac spinor} \text{ to } \psi = \begin{pmatrix} \chi \\ \bar{\phi} \end{pmatrix}, \tag{7.69}$$

$$\gamma_5 \equiv \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{\mathrm{i}}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \tag{7.70}$$

  $\Rightarrow$ Contruction of covariants: (see Exercise 9.3)

$$\overline{\psi}_1 \psi_2 = \text{scalar}, \tag{7.71}$$

$$\overline{\psi}_1 \gamma_5 \psi_2 = \text{pseudo-scalar}, \tag{7.72}$$

$$\overline{\psi}_1 \gamma^\mu \psi_2 = \text{vector}, \tag{7.73}$$

$$\overline{\psi}_1 \gamma^\mu \gamma_5 \psi_2 = \text{pseudo-vector}, \tag{7.74}$$

$$\overline{\psi}_1 \gamma^\mu \gamma^\nu \psi_2 = \text{rank-2 tensor}, \tag{7.75}$$

$$\overline{\psi}_1 \gamma^\mu \gamma^\nu \gamma_5 \psi_2 = \text{rank-2 pseudo-tensor}. \tag{7.76}$$

Some properties of the Dirac matrices: (see Exercise 9.2)

- $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \qquad \gamma_0\gamma^\mu\gamma_0 = (\gamma^\mu)^\dagger, \qquad \{\gamma^\mu, \gamma^5\} = 0,$ (7.77)
- $\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}, \qquad \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4[g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}],$ (7.78)

  $\mathrm{Tr}[\gamma^{\mu_1}\dots\gamma^{\mu_{2n+1}}] = 0, \quad n = 0, 1, \dots,$ (7.79)
- $\mathrm{Tr}[\gamma_5] = \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma_5] = 0, \qquad \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5] = -4i\epsilon^{\mu\nu\rho\sigma},$ (7.80)
- $\gamma^\alpha\gamma_\alpha = 4, \qquad \gamma^\alpha\gamma^\mu\gamma_\alpha = -2\gamma^\mu.$ (7.81)

7.5 Relativistic wave equations

7.5.1 Relativistic fields

Requirement by relativistic covariance:
Fields $\Phi_k(x)$ describing a specific particle transform in an *irreducible* representation of $L_+^\uparrow$ or $L_+^\uparrow \times P$ (reducible case: different irreducible blocks $\Rightarrow$ different particles).

Lorentz transformation: (Φ = classical field, no operator)

$$\Phi'_k(x') = \underbrace{S_{kl}(\Lambda)}_{\substack{\text{transformation matrix in irreducible}\\ \text{representation of } L_+^\uparrow \text{ or } L_+^\uparrow \times P}} \Phi_l(x), \qquad x' = \Lambda x. \tag{7.82}$$

$$\Phi'(x) = \underbrace{S(\Lambda)}_{\substack{\text{transformation of}\\ \text{inner degrees of}\\ \text{freedom} \to \text{ spin}}} \quad \Phi(\underbrace{\Lambda^{-1}x}_{\substack{\text{transformation of}\\ \text{space–time argument}\\ \hookrightarrow \text{ orbital angular momentum}}}) \tag{7.83}$$

Transformations in exponential form:

$$S(\Lambda) = \exp\Big\{-\frac{\mathrm{i}}{2}\omega_{\alpha\beta}M^{\alpha\beta}\Big\} = \text{finite-dim. representation}, \tag{7.84}$$

$$\Phi(\Lambda^{-1}x) = \underbrace{\exp\Big\{-\frac{\mathrm{i}}{2}\omega_{\alpha\beta}L^{\alpha\beta}\Big\}}_{\substack{\text{differential operator}\\ L^{\alpha\beta} = x^\alpha\hat{p}^\beta - x^\beta\hat{p}^\alpha\\ = \text{generalized orbital angular mom.}}} \Phi(x). \tag{7.85}$$

$$\Rightarrow \Phi'(x) = \exp\Big\{-\frac{\mathrm{i}}{2}\omega_{\alpha\beta}\underbrace{(M^{\alpha\beta} + L^{\alpha\beta})}_{\to \text{ total angular momentum}}\Big\} \Phi(x). \tag{7.86}$$

7.5.2 Relativistic wave equations for free particles

Basic requirements:

- Qm. superposition principle $\rightarrow$ *linearity* of differential equation
  General ansatz:
  $$\underbrace{\Pi_{kl}(m, \mathrm{i}\partial^\mu)}_{\substack{N\times N\text{-matrix-valued}\\ \text{differential operator}}} \Phi_l(x) = 0, \tag{7.87}$$
  where m = particle mass, N determined by particle spin.

  Order of differential eq. ≤ 2 (otherwise strange behaviour of solutions).

- *Covariance:* (7.87) has to imply
  $$0 \stackrel{!}{=} \Pi(m, \mathrm{i}\Lambda\partial)\,\Phi'(\Lambda x) \quad \text{with} \quad \Phi'(\Lambda x) = S(\Lambda)\,\Phi(x). \tag{7.88}$$
  $$\Rightarrow \Pi(m, \mathrm{i}\Lambda\partial) \stackrel{!}{=} \underbrace{S'(\Lambda)}_{\text{could also be in another representation}} \Pi(m, \mathrm{i}\partial)\,S(\Lambda^{-1}). \tag{7.89}$$

- *Mass-shell condition:*
  In momentum space only field modes with $p^2 = m^2$ should contribute to solution:
  $$\Phi(x) = \int \underbrace{\mathrm{d}\tilde{p}}_{= \frac{\mathrm{d}^4p}{(2\pi)^4}(2\pi)\delta(p^2-m^2)\,\theta(p_0)} \mathrm{e}^{-\mathrm{i}xp}\,\tilde{\Phi}(\vec{p}). \tag{7.90}$$

- *Spin projection*:
  Operator Π should project onto genuine spin-j part of representation $D^{(j_1,j_2)}$, where $|j_1 - j_2| \leq j \leq j_1 + j_2$. If Π does not perform this projection, additional constraints are needed to achieve it.

Examples:

- Klein–Gordon equation: $j = 0$.
  $$(\Box + m^2)\phi(x) = 0, \quad \Pi(m, \mathrm{i}\partial) = \partial^\mu\partial_\mu + m^2, \quad S(\Lambda) = 1, \tag{7.91}$$
  i.e. requirements trivially fulfilled.

- Maxwell equation for elmg. 4-vector potential A^μ in Lorenz gauge:
  $$\Box A^\mu = 0, \quad \Pi(\mathrm{i}\partial) = \partial^\mu\partial_\mu, \quad S(\Lambda) = \Lambda. \tag{7.92}$$
  Lorenz condition $\partial^\mu A_\mu = 0$ eliminates spin-0 part in $D^{(\frac{1}{2},\frac{1}{2})}$ representation, only spin-1 part remains.

  > Comment:
  > Photons do not really carry spin 1, but helicity $h = \pm 1$ (=spin projected to direction of flight), see below.

7.5.3 The Dirac equation

Wave equations for spin-$\frac{1}{2}$ fields:

Minimalistic attempt:

Wave equations for $\chi(x) \in D^{(\frac{1}{2},0)}$ and $\bar{\phi}(x) \in D^{(0,\frac{1}{2})}$ (smallest representations with $j = \frac{1}{2}$):

- Non-trivial transformation property of $\chi, \bar{\phi}$ should result from wave equation (otherwise additional constraints needed).
  $\hookrightarrow$ $\Pi(m, \mathrm{i}\partial)$ should mix field components (i.e. KG operator not acceptable).

- Relevant covariant objects for wave equations:

$$\chi,\ \bar{\sigma}^\mu \partial_\mu \bar{\phi} \ \in\ D^{(\frac{1}{2},0)}, \qquad \bar{\phi},\ \sigma^\mu \partial_\mu \chi \ \in\ D^{(0,\frac{1}{2})}. \tag{7.93}$$

  Proof:

$$\chi(x) \ \rightarrow\ \chi'(x') = \Lambda_R\, \chi(x) \tag{7.94}$$

$$\begin{aligned} \sigma^\mu \partial_\mu \chi(x) \ &\rightarrow\ \sigma^\mu \partial'_\mu \chi'(x') = \sigma^\mu \Lambda_\mu{}^\nu \partial_\nu\, \Lambda_R\, \chi(x) \\ &= \Lambda_L\, \underbrace{(\Lambda_L^{-1} \sigma^\mu \Lambda_R)}_{= \Lambda_R^\dagger \sigma^\mu \Lambda_R = \Lambda^\mu{}_\rho \sigma^\rho}\, \Lambda_\mu{}^\nu \partial_\nu\, \chi(x) = \Lambda_L\, \sigma^\mu \partial_\mu \chi(x). \end{aligned} \tag{7.95}$$

  $\bar{\sigma}^\mu \partial_\mu \bar{\phi}$ analogously. q.e.d.

- Consequence: Only possibility for separate wave equations for $\chi, \bar{\phi}$:

$$\sigma^\mu \partial_\mu \chi \ =\ 0, \qquad \bar{\sigma}^\mu \partial_\mu \bar{\phi} \ =\ 0. \qquad \textit{Weyl equations} \tag{7.96}$$

  Note: $\sigma^\mu \partial_\mu \chi = c\chi$, etc., not compatible with relativistic covariance !

- Solution of Weyl equations:

  Fourier ansatz: $\chi(x) = \mathrm{e}^{\pm \mathrm{i}kx}\, n_R$ with $n_R = \begin{pmatrix} n_{R,1} \\ n_{R,2} \end{pmatrix}$.

$$\hookrightarrow \quad 0 \ \overset{!}{=}\ \underbrace{\begin{pmatrix} k^0 - k^3 & -k^1 + \mathrm{i}k^2 \\ -k^1 - \mathrm{i}k^2 & k^0 + k^3 \end{pmatrix}}_{\det(...)\ = (k^0)^2 - (k^1)^2 - (k^2)^2 - (k^3)^2 = k^2} \begin{pmatrix} n_{R,1} \\ n_{R,2} \end{pmatrix}. \tag{7.97}$$

  $\Rightarrow$ Non-trivial solutions only for $k^2 = 0$, i.e. Weyl fermions are massless !

  Explicitly:

$$k^\mu = k^0(1, \vec{\mathrm{e}}), \quad \vec{\mathrm{e}} = \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{pmatrix} \qquad \Rightarrow \qquad n_R = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\varphi} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}. \tag{7.98}$$

  Analogously:

$$\bar{\phi}(x) = \mathrm{e}^{\pm \mathrm{i}kx}\, n_L, \quad k^2 = 0, \quad n_L = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\mathrm{e}^{\mathrm{i}\varphi} \cos\frac{\theta}{2} \end{pmatrix}. \tag{7.99}$$

The Dirac equation:

Combine Weyl spinors $\chi, \bar{\phi}$ to Dirac spinor $\psi = \begin{pmatrix} \chi \\ \bar{\phi} \end{pmatrix}$.

$\hookrightarrow$ Two covariant 1st-order equations possible:

$$\mathrm{i}\sigma^\mu \partial_\mu \chi \;=\; c_1 \bar{\phi}, \qquad \mathrm{i}\bar{\sigma}^\mu \partial_\mu \bar{\phi} \;=\; c_2 \chi. \tag{7.100}$$

Note: By appropriate rescaling, equality $c_1 = c_2 = m$ can be achieved.
(Identification of m as mass later.)

Matrix form:

$$\underbrace{\mathrm{i}\begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix}}_{=\gamma^\mu,\ \text{Dirac matrices}} \partial_\mu \begin{pmatrix} \chi \\ \bar{\phi} \end{pmatrix} - m \begin{pmatrix} \chi \\ \bar{\phi} \end{pmatrix} \;=\; 0. \tag{7.101}$$

$$\Rightarrow \quad (\mathrm{i}\gamma^\mu \partial_\mu - m)\,\psi \;=\; 0. \qquad \textit{Dirac equation} \tag{7.102}$$

Notation: $\not{a} \equiv \gamma^\mu a_\mu = \gamma_\mu a^\mu$ (*Feynman dagger*) $\Rightarrow$ Dirac eq.: $(\mathrm{i}\not{\partial} - m)\,\psi = 0$.

Covariance: (see Exercise 9.3)

$$\psi(x) \;\to\; \psi'(x') \;=\; S(\Lambda)\,\psi(x), \qquad S(\Lambda) \;=\; \begin{pmatrix} \Lambda_R & 0 \\ 0 & \Lambda_L \end{pmatrix}, \tag{7.103}$$

$$\not{a} \;\to\; \not{a}' \;=\; \gamma_\mu \Lambda^\mu{}_\nu a^\nu \;=\; S(\Lambda)\,\not{a}\,S(\Lambda)^{-1}. \tag{7.104}$$

$$\begin{aligned} \Rightarrow \quad (\mathrm{i}\not{\partial} - m)\,\psi(x) = 0 \;\to\; & (\mathrm{i}\not{\partial}' - m)\,\psi'(x') \\ & = S(\Lambda)\,(\mathrm{i}\not{\partial} - m)\,S(\Lambda)^{-1}\,S(\Lambda)\,\psi(x) \\ & = S(\Lambda)\,(\mathrm{i}\not{\partial} - m)\,\psi(x) = 0. \end{aligned} \tag{7.105}$$

Chapter 8

Free Dirac fermions

8.1 Solutions of the classical Dirac equation

Dirac eq.: $(\mathrm{i}\partial\!\!\!/ - m)\,\psi = 0.$

Note: Each component of ψ obeys the KG equation:

$$0 \;=\; (-\mathrm{i}\partial\!\!\!/ - m)\,(\mathrm{i}\partial\!\!\!/ - m)\,\psi \;=\; (\underbrace{\partial\!\!\!/^2}_{=\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu = g^{\mu\nu}\partial_\mu\partial_\nu = \partial^2} + m^2)\,\psi \;=\; (\Box + m^2)\,\psi. \tag{8.1}$$

Fourier ansatz:

$$\psi(x) = \mathrm{e}^{-\mathrm{i}kx}\,u(k), \quad u(k) = \text{constant 4-component Dirac spinor.} \tag{8.2}$$

- Eq. (8.1) implies $k^2 = m^2$. $\to$ Set $k_0 = \sqrt{\vec{k}^{\,2} + m^2}$.
- Ansatz leads to Dirac eq. in momentum space:

$$(k\!\!\!/ - m)\,u(k) \;=\; 0. \tag{8.3}$$

$\hookrightarrow$ 4-dim. system of linear equations.

Solution of Eq. (8.3) **in two steps:**

1. Solve equation first in rest frame of k^μ, i.e. for $k_r^\mu = (m, \vec{0})$:

$$k^\mu \;=\; (k_0, \vec{k}) \;=\; \Lambda^\mu{}_\nu k_r^\nu, \quad \vec{k} \;=\; |\vec{k}|\,\vec{\mathrm{e}}, \quad \vec{\mathrm{e}} \;=\; \begin{pmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{pmatrix}. \tag{8.4}$$

Using the chiral representation for γ^μ yields

$$0 \;=\; (k\!\!\!/_r - m)\,u(k_r) \;=\; m(\gamma_0 - \mathbf{1})\,u(k_r) \;=\; m\begin{pmatrix} -\mathbf{1} & +\mathbf{1} \\ +\mathbf{1} & -\mathbf{1} \end{pmatrix} u(k_r). \tag{8.5}$$

$\hookrightarrow$ Two independent solutions of the block form $u(k_r) = \begin{pmatrix} n \\ n \end{pmatrix}$, e.g. $n = n_{R,L}$.

2. Boost k_r^μ into original system:

$$u(k) = S(\Lambda)\,u(k_r) = \begin{pmatrix}\Lambda_R & 0\\ 0 & \Lambda_L\end{pmatrix} u(k_r), \tag{8.6}$$

with $\quad \Lambda_R = \exp\Big\{\frac{\nu}{2}\vec{e}\,\vec{\sigma}\Big\}, \quad \Lambda_L = \exp\Big\{-\frac{\nu}{2}\vec{e}\,\vec{\sigma}\Big\}, \quad \nu = \frac{1}{2}\ln\left(\frac{k_0+|\vec{k}|}{k_0-|\vec{k}|}\right) = \text{rapidity}.$

Simple form of $\Lambda_{R,L}$ after diagonalizing $\vec{e}\,\vec{\sigma} = U\sigma^3 U^\dagger$:

$$\Lambda_R = U\begin{pmatrix}e^{\nu/2} & 0\\ 0 & e^{-\nu/2}\end{pmatrix}U^\dagger, \qquad \Lambda_L = U\begin{pmatrix}e^{-\nu/2} & 0\\ 0 & e^{\nu/2}\end{pmatrix}U^\dagger, \tag{8.7}$$

$$\text{with} \qquad U = (n_R, n_L) = \begin{pmatrix}e^{-i\varphi}\cos\frac{\theta}{2} & \sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & -e^{i\varphi}\cos\frac{\theta}{2}\end{pmatrix}, \qquad e^{\pm\nu/2} = \sqrt{\frac{k_0 \pm |\vec{k}|}{m}},$$

with $n_{R,L}$ defined in Eqs. (7.98) and (7.99).

Using $U^\dagger n_R = \begin{pmatrix}1\\0\end{pmatrix}$ and $U^\dagger n_L = \begin{pmatrix}0\\1\end{pmatrix}$ we obtain:

$$\begin{aligned}\Lambda_R\,n_R &= e^{+\nu/2}\,n_R, & \Lambda_R\,n_L &= e^{-\nu/2}\,n_L,\\ \Lambda_L\,n_R &= e^{-\nu/2}\,n_R, & \Lambda_L\,n_L &= e^{+\nu/2}\,n_L.\end{aligned} \tag{8.8}$$

$\Rightarrow\;$ $n \propto n_{R,L}$ is convenient choice for n in $u(k_r)$ of step 1.

Two independent standard solutions:

$$\begin{aligned}u_R(k) &= S(\Lambda)\sqrt{m}\begin{pmatrix}n_R\\ n_R\end{pmatrix} = \sqrt{m}\begin{pmatrix}\Lambda_R\,n_R\\ \Lambda_L\,n_R\end{pmatrix} = \begin{pmatrix}\sqrt{k_0+|\vec{k}|}\,n_R\\ \sqrt{k_0-|\vec{k}|}\,n_R\end{pmatrix},\\ u_L(k) &= S(\Lambda)\sqrt{m}\begin{pmatrix}n_L\\ n_L\end{pmatrix} = \sqrt{m}\begin{pmatrix}\Lambda_R\,n_L\\ \Lambda_L\,n_L\end{pmatrix} = \begin{pmatrix}\sqrt{k_0-|\vec{k}|}\,n_L\\ \sqrt{k_0+|\vec{k}|}\,n_L\end{pmatrix}.\end{aligned} \tag{8.9}$$

Analogous procedure for ansatz $\psi(x) = e^{+ikx}\,v(k)$ leads to $(\not{k}+m)\,v = 0$ with the standard solutions:

$$v_R(k) = \begin{pmatrix}\sqrt{k_0+|\vec{k}|}\,n_R\\ -\sqrt{k_0-|\vec{k}|}\,n_R\end{pmatrix}, \qquad v_L(k) = \begin{pmatrix}-\sqrt{k_0-|\vec{k}|}\,n_L\\ \sqrt{k_0+|\vec{k}|}\,n_L\end{pmatrix}. \tag{8.10}$$

Normalization: $\quad (\overline{u} \equiv u^\dagger\gamma_0)$

$$\begin{aligned}\overline{u}_\sigma(k)\,u_\tau(k) &= 2m\,\delta_{\sigma\tau}, \qquad \sigma,\tau = R,L,\\ \overline{u}_\sigma(k)\,v_\tau(k) &= 0,\\ \overline{v}_\sigma(k)\,u_\tau(k) &= 0,\\ \overline{v}_\sigma(k)\,v_\tau(k) &= -2m\,\delta_{\sigma\tau},\end{aligned} \tag{8.11}$$

Spin orientations of the solutions:

Spin operator in Dirac representation $D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$:

$$\vec{S} = \vec{J}_{(\frac{1}{2},0)} \oplus \vec{J}_{(0,\frac{1}{2})} = \frac{\vec{\sigma}}{2} \oplus \frac{\vec{\sigma}}{2} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}. \tag{8.12}$$

Definition: *Helicity* = spin projection onto direction of flight $\vec{\mathrm{e}}$

$$h \;\equiv\; \vec{\mathrm{e}}\,\vec{S} \;=\; \frac{1}{2}\begin{pmatrix} \vec{\mathrm{e}}\,\vec{\sigma} & 0 \\ 0 & \vec{\mathrm{e}}\,\vec{\sigma} \end{pmatrix}. \tag{8.13}$$

Standard solutions $u_{R,L}$ and $v_{R,L}$ are helicity eigenstates:

$$h\,u_R = +\frac{1}{2}u_R, \qquad h\,u_L = -\frac{1}{2}u_L, \qquad h\,v_R = +\frac{1}{2}v_R, \qquad h\,v_L = -\frac{1}{2}v_L. \tag{8.14}$$

$\Rightarrow$ Particle solutions $\psi_{+,R/L}(x) = \mathrm{e}^{-\mathrm{i}kx}\,u_{R/L}(k)$ correspond to helicity states with $h = \pm\frac{1}{2}$.
Note: Helicity content of antiparticle solutions $\psi_{-,R/L}(x) = \mathrm{e}^{+\mathrm{i}kx}\,v_{R/L}(k)$ clarified by QFT.

General solution of free (classical) Dirac equation:

$$\psi(x) \;=\; \int \mathrm{d}\tilde{k} \sum_{\sigma=R,L} \Big[\underbrace{a_\sigma(\vec{k})}\, u_\sigma(k)\mathrm{e}^{-\mathrm{i}kx} + \underbrace{b^*_\sigma(\vec{k})}\, v_\sigma(k)\mathrm{e}^{+\mathrm{i}kx} \Big]. \tag{8.15}$$

arbitrary functions of $\vec{k}$
$\hookrightarrow$ creation/annihilation operators for (anti)particles in QFT

8.2 Quantization of free Dirac fields

8.2.1 Quantization procedure

Classical Lagrangian:

Dirac equation obviously reproduced by

$$\mathcal{L} = \overline{\psi}\,(\mathrm{i}\partial\!\!\!/ - m)\,\psi, \tag{8.16}$$

considering ψ and $\overline{\psi}$ as independent.

Euler–Lagrange equations:

$$0 = \frac{\partial\mathcal{L}}{\partial\overline{\psi}} - \partial^\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\overline{\psi})}\right) = \frac{\partial\mathcal{L}}{\partial\overline{\psi}} = (\mathrm{i}\partial\!\!\!/ - m)\,\psi, \tag{8.17}$$

$$0 = \frac{\partial\mathcal{L}}{\partial\psi} - \partial^\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\psi)}\right) = -m\overline{\psi} - \partial^\mu\left(\mathrm{i}\overline{\psi}\gamma_\mu\right) = -\overline{\psi}\left(m + \mathrm{i}\overleftarrow{\partial\!\!\!/}\right) \quad \textit{adjoint Dirac eq..} \tag{8.18}$$

Canonical commutators:

Preliminary consideration:
Commutators $[a(\vec{k}), a^\dagger(\vec{p})]$, etc., lead to totally symmetric states $|\vec{k},\vec{p}\rangle = a^\dagger(\vec{p})a^\dagger(\vec{p})\,|0\rangle$.
But: Fermions require totally antisymmetric states.
$\hookrightarrow$ Use ansatz with anticommutators $\{a(\vec{k}), a^\dagger(\vec{p})\}$ in quantization !

Canonical momentum variable to field $\psi(x)$:

$$\pi(x) = -\frac{\partial\mathcal{L}}{\partial(\partial^0\psi)} = \mathrm{i}\overline{\psi}\gamma_0 = \mathrm{i}\psi^\dagger. \qquad \text{(sign change owing to anticommutativity of } \psi) \tag{8.19}$$

Canonical equal-time anticommutators for quantization:

$$\begin{aligned}
\left\{\psi_\alpha(t,\vec{x}), \mathrm{i}\psi^\dagger_\beta(t,\vec{y})\right\} &= \mathrm{i}\,\mathbf{1}\,\delta_{\alpha\beta}\delta(\vec{x}-\vec{y}), \quad \text{i.e.} \quad \left\{\psi_\alpha(t,\vec{x}), \overline{\psi}_\beta(t,\vec{y})\right\} = (\gamma_0)_{\alpha\beta}\,\delta(\vec{x}-\vec{y}),\\
\left\{\psi_\alpha(t,\vec{x}), \psi_\beta(t,\vec{y})\right\} &= \left\{\overline{\psi}_\alpha(t,\vec{x}), \overline{\psi}_\beta(t,\vec{y})\right\} = 0,
\end{aligned} \tag{8.20}$$

Insertion of Fourier decompositions

$$\psi(x) = \int \mathrm{d}\tilde{k} \sum_{\sigma=R,L}\left[\, a_\sigma(\vec{k})u_\sigma(k)\mathrm{e}^{-\mathrm{i}kx} + b^\dagger_\sigma(\vec{k})v_\sigma(k)\mathrm{e}^{+\mathrm{i}kx}\,\right], \tag{8.21}$$

$$\overline{\psi}(x) = \int \mathrm{d}\tilde{k} \sum_{\sigma=R,L}\left[\, a^\dagger_\sigma(\vec{k})\overline{u}_\sigma(k)\mathrm{e}^{+\mathrm{i}kx} + b_\sigma(\vec{k})\overline{v}_\sigma(k)\mathrm{e}^{-\mathrm{i}kx}\,\right] \tag{8.22}$$

yields anticommutators for creation/annihilation operators:

$$\begin{aligned}
\left\{a_\sigma(\vec{k}), a^\dagger_\tau(\vec{p})\right\} &= \left\{b_\sigma(\vec{k}), b^\dagger_\tau(\vec{p})\right\} = (2\pi)^3 2p_0\,\delta(\vec{k}-\vec{p})\,\delta_{\sigma\tau},\\
\left\{a_\sigma(\vec{k}), a_\tau(\vec{p})\right\} &= 0, \quad \text{etc.}
\end{aligned} \tag{8.23}$$

8.2.2 Particle states and Fock space

Fock space:

- Ground state $|0\rangle$ (*vacuum*, no particle excitation): $\langle 0| = (|0\rangle)^\dagger$, $\langle 0|0\rangle = 1$.

$$a_\sigma(\vec{p})\,|0\rangle = 0\,, \quad b_\sigma(\vec{p})\,|0\rangle = 0 \quad \forall\,\vec{p}. \tag{8.24}$$

- Excited states (particle states):

$$\begin{aligned}
|f_\sigma(\vec{p}_1)\rangle &= a_\sigma^\dagger(\vec{p}_1)\,|0\rangle && \text{1 fermion} && (8.25)\\
|\bar{f}_\sigma(\vec{p}_1)\rangle &= b_\sigma^\dagger(\vec{p}_1)\,|0\rangle && \text{1 antifermion} && (8.26)\\
|f_\sigma(\vec{p}_1)f_\tau(\vec{p}_2)\rangle &= a_\sigma^\dagger(\vec{p}_1)a_\tau^\dagger(\vec{p}_2)\,|0\rangle && \text{2 fermions} && (8.27)\\
&= -a_\tau^\dagger(\vec{p}_2)a_\sigma^\dagger(\vec{p}_1)\,|0\rangle && && (8.28)\\
&= -\,|f_\tau(\vec{p}_2)f_\sigma(\vec{p}_1)\rangle && && (8.29)\\
&[= 0 \quad \text{if } \vec{p}_1 = \vec{p}_2 \text{ and } \sigma = \tau]\\
&\vdots
\end{aligned}$$

  Antisymmetric states $\Rightarrow$ *Fermi–Dirac statistics*

- *Fock space* = Hilbert space spanned by all fermion and antifermion states:

$$\left\{\,|0\rangle\,, |f_\sigma(\vec{p}_1)\rangle\,, |\bar{f}_\tau(\vec{p}_2)\rangle\,, |f_\sigma(\vec{p}_1)f_\tau(\vec{p}_2)\rangle \ldots\right\}$$

- *(Anti)Fermion number operators:*

$$\begin{aligned}
N_{f_\sigma}(\vec{p}) &= a_\sigma^\dagger(\vec{p})a_\sigma(\vec{p}), & N_f &= \sum_{\sigma=R,L}\int \mathrm{d}\tilde{p}\, N_{f_\sigma}(\vec{p}),\\
N_{\bar{f}_\sigma}(\vec{p}) &= b_\sigma^\dagger(\vec{p})b_\sigma(\vec{p}), & N_{\bar{f}} &= \sum_{\sigma=R,L}\int \mathrm{d}\tilde{p}\, N_{\bar{f}_\sigma}(\vec{p}).
\end{aligned} \tag{8.30}$$

  $\hookrightarrow$ Commutator relations:

$$\begin{aligned}
[N_f, a_\sigma^\dagger(\vec{p})] &= +a_\sigma^\dagger(\vec{p}), & [N_f, a_\sigma(\vec{p})] &= -a_\sigma(\vec{p}),\\
[N_f, b_\sigma^\dagger(\vec{p})] &= 0, & [N_f, b_\sigma(\vec{p})] &= 0,
\end{aligned} \tag{8.31}$$

$$\text{analogously for } N_{\bar{f}} \text{ with } a \leftrightarrow b.$$

Field operators and wave functions: (cf. scalar fields, Sect. 5.4)

One-particle wave function $\varphi(x)$ corresponding to fermion state $|f_\sigma(\vec{p})\rangle$:

$$\varphi_{f_\sigma(\vec{p})}(x) \equiv \langle 0|\psi(x)|f_\sigma(\vec{p})\rangle = \langle 0|\psi(x)\, a^\dagger_\sigma(\vec{p})|0\rangle = \mathrm{e}^{-\mathrm{i}px}\, u_\sigma(p). \tag{8.32}$$

Space–time transformations of $|f\rangle$, $\psi(x)$, and $\varphi(x)$: $x \to x' = \Lambda x + a$

- Qm. states:

$$|f\rangle \to |f'\rangle = U(\Lambda, a)\,|f\rangle \quad \text{with } U = \text{unitary operator.} \tag{8.33}$$

$\hookrightarrow$ Transition amplitudes $\langle f'|g'\rangle = \langle f|U^\dagger U|g\rangle = \langle f|g\rangle =$ invariant.

> Comment:
> $U(\Lambda, a)$ are transformations in ∞-dimensional representation of the Poincaré group, which is spanned by the particle states.

- Field operator:

$$\psi(x') = U(\Lambda, a)\, \underbrace{S(\Lambda)}_{\text{transformation of spin part of } \psi(x)}\, \psi(x)\, U^\dagger(\Lambda, a), \tag{8.34}$$

so that scalar products $\langle f|...\overline{\psi}_1(x)...\psi_2(x)...|g\rangle = \langle f'|...\overline{\psi}_1(x')...\psi_2(x')...|g'\rangle =$ invariant.

$$\Rightarrow\; U(\Lambda, a)^\dagger\, \psi(x)\, U(\Lambda, a) = S(\Lambda)\, \psi(\Lambda^{-1}(x-a)). \tag{8.35}$$

- Wave function:

$$\begin{aligned} \varphi'(x') = \langle 0|\psi(x')|f'\rangle &= \underbrace{\langle 0|\, U(\Lambda, a)}_{=\langle 0| \,=\, \text{invariant}}\; S(\Lambda)\; \psi(x)\; \underbrace{U^\dagger(\Lambda, a) U(\Lambda, a)}_{=\mathbf{1}}\; |f\rangle \\ &= S(\Lambda)\; \langle 0|\psi(x)|f\rangle = S(\Lambda)\, \varphi(x). \end{aligned} \tag{8.36}$$

$$\Rightarrow\; \varphi'(x) = S(\Lambda)\, \varphi\left(\Lambda^{-1}(x-a)\right). \tag{8.37}$$

Wave functins transform like classical fields.

Properties of the particle states:

- Electric charge:

  Electric current density: (= Noether current for symmetry $\psi \to \psi' = \mathrm{e}^{-\mathrm{i}q\omega}\psi$)

$$j^\mu = q\overline{\psi}\gamma^\mu\psi. \tag{8.38}$$

  $\Rightarrow$ Operator Q for conserved electric charge:

$$Q = q\int \mathrm{d}^3x\ :\ \bar{\psi}\gamma^0\psi\ := q\int \mathrm{d}\tilde{p}\sum_{\sigma=R,L}\left[a^\dagger_\sigma(\vec{p})a_\sigma(\vec{p}) - b^\dagger_\sigma(\vec{p})b_\sigma(\vec{p})\right] = q(N_f - N_{\bar{f}}). \tag{8.39}$$

  Charges of particle states:

$$\begin{aligned} Q\,|f_\sigma(\vec{p})\rangle &= q(N_f - N_{\bar{f}})a^\dagger_\sigma(\vec{p})\,|0\rangle \\ &= q\Big(\underbrace{[N_f, a^\dagger_\sigma(\vec{p})]}_{=a^\dagger_\sigma(\vec{p})} - \underbrace{[N_{\bar{f}}, a^\dagger_\sigma(\vec{p})]}_{=0}\Big)\,|0\rangle = q\,|f_\sigma(\vec{p})\rangle\,, \end{aligned} \tag{8.40}$$

$$Q\,|\bar{f}_\sigma(\vec{p})\rangle = \cdots = -q\,|\bar{f}_\sigma(\vec{p})\rangle\,. \tag{8.41}$$

  $\Rightarrow$ Fermion f carries charge $+q$, antifermion $\bar{f}$ charge $-q$.

- 4-momentum:

  Energy-momentum tensor: (derived as for scalar fields)

$$\theta^{\mu\nu} = \frac{\mathrm{i}}{2}\,\overline{\psi}\,\overleftrightarrow{\partial^\nu}\,\gamma^\mu\psi. \tag{8.42}$$

  $\Rightarrow$ Operator P^μ for 4-momentum:

$$P^\mu = \int \mathrm{d}^3x\ :\ \theta^{0\mu}\ := \cdots = \int \mathrm{d}\tilde{p}\,p^\mu\sum_\sigma\left[N_{f_\sigma}(\vec{p}) + N_{\bar{f}_\sigma}(\vec{p})\right]. \tag{8.43}$$

$$\hookrightarrow\ \left[P^\mu, a^\dagger_\sigma(\vec{p})\right] = p^\mu a^\dagger_\sigma(\vec{p})\,,\quad \left[P^\mu, b^\dagger_\sigma(\vec{p})\right] = p^\mu b^\dagger_\sigma(\vec{p})\,. \tag{8.44}$$

  4-momenta of the particle states:

$$P^\mu\,|f_\sigma(\vec{p})\rangle = [P^\mu, a^\dagger_\sigma(\vec{p})]\,|0\rangle = p^\mu a^\dagger_\sigma(\vec{p})\,|0\rangle = p^\mu\,|f_\sigma(\vec{p})\rangle\,, \tag{8.45}$$

$$P^\mu\,|\bar{f}_\sigma(\vec{p})\rangle = \cdots = p^\mu\,|\bar{f}_\sigma(\vec{p})\rangle\,. \tag{8.46}$$

  $\Rightarrow$ Both elementary fermion and antifermion states carry 4-momentum p^μ.

  Alternative derivation of Eq. (8.44) via translation property of operator $\psi(x)$:

$$\underbrace{U(\Lambda = \mathbf{1}, a)^\dagger}_{=\exp\{-\mathrm{i}P_\mu a^\mu\}}\ \psi(x)\ \ \underbrace{U(\mathbf{1}, a)}_{=\exp\{+\mathrm{i}P_\mu a^\mu\}}\ =\ \psi(x-a). \tag{8.47}$$

  $\hookrightarrow$ Taking a^μ infinitesimal yields

$$[P_\mu, \psi(x)]\ =\ -\mathrm{i}\partial_\mu\psi(x). \tag{8.48}$$

  $\hookrightarrow$ Commutators $[P^\mu, a^{(\dagger)}_\sigma(\vec{p})]$, etc. from plane-wave solution for $\psi(x)$.

- Spin and helicity:

  Make use of Lorentz transformation property of ψ:

$$U(\Lambda, a=0)^\dagger\, \psi(x)\, U(\Lambda, 0) = S(\Lambda)\, \psi(\Lambda^{-1}x), \tag{8.49}$$

  where

  - $\diamond$ $U(\Lambda, 0) = \exp\{-\frac{\mathrm{i}}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\}$,
    with $\mathcal{J}^{\mu\nu}$ = abstract operator of generalized total angular momentum,
  - $\diamond$ $S(\Lambda) = \begin{pmatrix} \Lambda_R & 0 \\ 0 & \Lambda_L \end{pmatrix}$ = spin transformation matrix in Dirac representation, with generators $M^{\mu\nu}$.

$$\vec{S} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} = \text{ spin part of } M^{\mu\nu}\text{, see Eq. (8.12).} \tag{8.50}$$

  - $\diamond$ $\psi(\Lambda^{-1}x) = \exp\{-\frac{\mathrm{i}}{2}\omega_{\mu\nu}L^{\mu\nu}\}\, \psi(x)$,
    where $L^{\mu\nu} = \mathrm{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)$ = generalized orbital angular momentum.

  Connection between $\mathcal{J}^{\mu\nu}$, $M^{\mu\nu}$, and $L^{\mu\nu}$ derived upon taking $\omega_{\mu\nu}$ infinitesimal:

$$[\mathcal{J}^{\mu\nu}, \psi(x)] = -\left(M^{\mu\nu} + L^{\mu\nu}\right)\psi(x). \tag{8.51}$$

  Restriction to rotational part of spin transformation:

$$\left[\vec{J}, \psi(x)\right] = -\vec{S}\,\psi(x), \quad \text{where } \vec{J} = \text{abstract generator for spin rotations.} \tag{8.52}$$

  $\Rightarrow$ Helicities ($\vec{\mathrm{e}} = \vec{p}/|\vec{p}|$ directions of $\vec{J}$, $\vec{S}$) of Fock states:

$$\begin{aligned}
\left[\vec{\mathrm{e}}\cdot\vec{J}, a_\sigma(\vec{p})\right] &= \left[\vec{\mathrm{e}}\cdot\vec{J}, \int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}px}\, u_\sigma^\dagger(p)\,\psi(x)\right] \\
&= \int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}px}\, u_\sigma^\dagger(p)\, \underbrace{\vec{\mathrm{e}}\cdot\left[\vec{J}, \psi(x)\right]}_{= -\vec{\mathrm{e}}\cdot\vec{S}\,\psi(x) = -h\,\psi(x)} \\
&= -\int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}px}\, \underbrace{u_\sigma^\dagger(p)\, h}_{\substack{= u_\sigma^\dagger(p)\, h^\dagger = [h\, u_\sigma(p)]^\dagger = \frac{1}{2}\mathrm{sgn}(\sigma) u_\sigma(p)^\dagger, \\ \text{where } \mathrm{sgn}(R/L) = +/-\text{, see Eq. (8.14)}}}\, \psi(x) \\
&= -\frac{1}{2}\mathrm{sgn}(\sigma) \int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}px}\, u_\sigma^\dagger(p)\,\psi(x) \\
&= -\frac{1}{2}\mathrm{sgn}(\sigma)\, a_\sigma(\vec{p}),
\end{aligned} \tag{8.53}$$

$$
\begin{aligned}
\left[\vec{\mathrm{e}}\cdot\vec{J}, b_\sigma^\dagger(\vec{p})\right] &= \left[\vec{\mathrm{e}}\cdot\vec{J}, \int \mathrm{d}^3x\, \mathrm{e}^{-\mathrm{i}px}\, v_\sigma^\dagger(p)\, \psi(x)\right] \\
&= \int \mathrm{d}^3x\, \mathrm{e}^{-\mathrm{i}px}\, v_\sigma^\dagger(p)\, \vec{\mathrm{e}}\cdot\left[\vec{J}, \psi(x)\right] \\
&= -\int \mathrm{d}^3x\, \mathrm{e}^{-\mathrm{i}px}\, \underbrace{v_\sigma^\dagger(p)\, h}_{=\,[h\, v_\sigma(p)]^\dagger \,=\, \frac{1}{2}\mathrm{sgn}(\sigma) v_\sigma(p)^\dagger,\ \text{see Eq. (8.14)}}\, \psi(x) \\
&= -\frac{1}{2}\mathrm{sgn}(\sigma)\, b_\sigma^\dagger(\vec{p}),
\end{aligned}
\tag{8.54}
$$

$$
\hookrightarrow\ \left[\vec{\mathrm{e}}\cdot\vec{J}, a_\sigma^\dagger(\vec{p})\right] = +\frac{1}{2}\mathrm{sgn}(\sigma)\, a_\sigma^\dagger(\vec{p}), \qquad \left[\vec{\mathrm{e}}\cdot\vec{J}, b_\sigma(\vec{p})\right] = +\frac{1}{2}\mathrm{sgn}(\sigma)\, b_\sigma(\vec{p}). \tag{8.55}
$$

$$
\begin{aligned}
\Rightarrow\ \vec{\mathrm{e}}\cdot\vec{J}\, |f_\sigma(\vec{p})\rangle &= \left[\vec{\mathrm{e}}\cdot\vec{J}, a_\sigma^\dagger(\vec{p})\right]\, |0\rangle = +\frac{1}{2}\mathrm{sgn}(\sigma)\, a_\sigma^\dagger(\vec{p})\, |0\rangle = +\frac{1}{2}\mathrm{sgn}(\sigma)\, |f_\sigma(\vec{p})\rangle\,, \\
\vec{\mathrm{e}}\cdot\vec{J}\, |\bar{f}_\sigma(\vec{p})\rangle &= \ldots = -\frac{1}{2}\mathrm{sgn}(\sigma)\, |\bar{f}_\sigma(\vec{p})\rangle\,,
\end{aligned}
\tag{8.56}
$$

i.e. fermion state $|f_{R/L}\rangle$ has helicity $\pm\frac{1}{2}$,
but antifermion state $|\bar{f}_{R/L}\rangle$ has helicity $\mp\frac{1}{2}$.

8.2.3 Fermion propagator

Definition:

$$
\langle 0|\, T\psi(x)\bar{\psi}(y)\, |0\rangle = \mathrm{i}S_F(x,y) \tag{8.57}
$$

fermion propagator

Note: Each (anti)commutation of two fermionic operators in : ... : and $T(\ldots)$ products leads to a sign change !

Calculation of $S_F(x,y)$:

$$
\begin{aligned}
\langle 0|\, T\psi(x)\overline{\psi}(y)\, |0\rangle \;=\;& \langle 0|\, T \int \mathrm{d}\tilde{k} \sum_\sigma \Big[\, a_\sigma(\vec{k}) u_\sigma(k) \mathrm{e}^{-\mathrm{i}kx} + b_\sigma^\dagger(\vec{k}) v_\sigma(k) \mathrm{e}^{+\mathrm{i}kx} \,\Big] \\
& \times \int \mathrm{d}\tilde{p} \sum_\tau \Big[\, a_\tau^\dagger(\vec{p}) \overline{u}_\tau(p) \mathrm{e}^{+\mathrm{i}py} + b_\tau(\vec{p}) \overline{v}_\tau(p) \mathrm{e}^{-\mathrm{i}py} \,\Big]\, |0\rangle \\
=\;& \theta(x_0 - y_0)\, \langle 0| \int \mathrm{d}\tilde{k} \int \mathrm{d}\tilde{p} \sum_{\sigma,\tau} \mathrm{e}^{-\mathrm{i}(kx-py)} u_\sigma(k) \overline{u}_\tau(p) a_\sigma(\vec{k}) a_\tau^\dagger(\vec{p})\, |0\rangle \\
& \qquad \hookrightarrow\ \text{fermion propagation from } y \text{ to } x \\
& - \theta(y_0 - x_0)\, \langle 0| \int \mathrm{d}\tilde{k} \int \mathrm{d}\tilde{p} \sum_{\sigma,\tau} \mathrm{e}^{\mathrm{i}(kx-py)} v_\sigma(k) \overline{v}_\tau(p) b_\tau(\vec{p}) b_\sigma^\dagger(\vec{k})\, |0\rangle \\
& \qquad \hookrightarrow\ \text{antifermion propagation from } x \text{ to } y
\end{aligned}
\tag{8.58}
$$

Using $\langle 0| a_\sigma(\vec{k}) a_\tau^\dagger(\vec{p}) |0\rangle = (2\pi)^3 2k^0 \delta(\vec{k}-\vec{p})\delta_{\sigma\tau}$, etc., this yields

$$\begin{aligned}
\langle 0| T\psi(x)\overline{\psi}(y) |0\rangle = \quad & \theta(x_0-y_0)\int \mathrm{d}\tilde{k}\, \mathrm{e}^{-\mathrm{i}k(x-y)} \underbrace{\sum_\sigma u_\sigma(k)\overline{u}_\sigma(k)}_{=\not{k}+m,\quad \textit{completeness relation, } \text{see Exercise 10.1}} \\
& - \theta(y_0-x_0)\int \mathrm{d}\tilde{k}\, \mathrm{e}^{\mathrm{i}k(x-y)} \underbrace{\sum_\sigma v_\sigma(k)\overline{v}_\sigma(k)}_{=\not{k}-m} \\
&= \theta(x_0-y_0)\int \mathrm{d}\tilde{k}\, \mathrm{e}^{-\mathrm{i}k(x-y)}\,(\not{k}+m) - \theta(y_0-x_0)\int \mathrm{d}\tilde{k}\, \mathrm{e}^{\mathrm{i}k(x-y)}\,(\not{k}-m) \\
&= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}k(x-y)}\,\frac{\mathrm{i}(\not{k}+m)}{k^2-m^2+\mathrm{i}\epsilon}, \qquad \text{as for scalar propagator, see Sect. 4.3.2} \\
&= (\mathrm{i}\not{\partial}_x+m)\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}k(x-y)}}{k^2-m^2+\mathrm{i}\epsilon} = (\mathrm{i}\not{\partial}_x+m)\,\mathrm{i}D_F(x,y). \qquad (8.59)
\end{aligned}$$

Consequences of $S_F(x,y) = (\mathrm{i}\not{\partial}_x+m)\,D_F(x,y)$:

- $S_F(x,y)$ has the same causal properties as scalar propagator $D_F(x,y)$.
- Differential equation:

$$\begin{aligned}
(\mathrm{i}\not{\partial}_x-m)S_F(x,y) &= (\mathrm{i}\not{\partial}_x-m)(\mathrm{i}\not{\partial}_x+m)\,D_F(x,y) = -(\Box_x+m^2)\,D_F(x,y) \\
&= \delta(x-y). \qquad (8.60)
\end{aligned}$$

  $S_F(x,y)$ is inverse of the Dirac operator $(\mathrm{i}\not{\partial}-m)$.

8.2.4 Connection between spins and statistics

Spin statistics theorem:

- Fields with integer spin (0, 1, ...) are quantized with commutators.
  ↪ States obey Bose–Einstein statistics.
- Fields with half-integer spin (1/2, 3/2, ...) are quantized with anticommutatorns.
  ↪ States obey Fermi–Dirac statistics.

"Proof": otherwise several inconsistencies:

- Violation of causality, i.e. violation of

$$[\mathrm{Obs}(x), \mathrm{Obs}(y)] \;=\; 0 \quad \text{for } (x-y)^2 < 0. \qquad (8.61)$$

- Energy spectrum not bounded from below. $\rightarrow$ System unstable.
- Statement on relation between spin and BE / FD statistics supported by experiment.

Chapter 9

Interaction of scalar and fermion fields

9.1 Interacting fermion fields

Interaction Lagrangians with a Dirac fermion:

$$\mathcal{L} = \overline{\psi}\,(\mathrm{i}\partial\!\!\!/ - m)\,\psi - V(\psi, \overline{\psi}, \Phi). \tag{9.1}$$

Properties of interaction potential V:

- Each term in V contains products of at least 3 fields (2 fields $\to$ free propagation).
- V = Lorentz invariant.
  $\hookrightarrow$ ψ always appears in products $\overline{\psi}...\psi$.
- V has mass dimension 4.
  (Fields: $\dim[\phi] = \dim[A^\mu] = 1$, $\dim[\psi] = \frac{3}{2}$.)
- $V = V^\dagger$ = hermitian.

Examples:

- Yukawa interaction of a fermion and a scalar field ϕ:

$$V = y\,\phi\,\overline{\psi}\,\psi, \quad y = \text{dimensionless coupling strength.} \tag{9.2}$$

  $\hookrightarrow$ Basic interaction between fermions and the Higgs boson in the Standard Model.

- Electromagnetic interaction:

$$V = Qe\overline{\psi}\,A\!\!\!/\,\psi, \quad e = \text{elementary charge, } Q = \text{relative fermion charge.} \tag{9.3}$$

Comments on the perturbative machinery:

$\hookrightarrow$ works as for scalar fields with few exceptions (signs!):

- EOMs: $\dfrac{\delta\mathcal{L}}{\delta\psi} = 0, \quad \dfrac{\delta\mathcal{L}}{\delta\overline{\psi}} = 0.$

  $\hookrightarrow$ Operators $\dfrac{\partial}{\partial\psi}$, etc., anticommute with fermionic fields.

- Symmetries if fields Φ_k are bosonic or fermionic:

$$\Phi_k \to \Phi_k + \delta\omega_a \Delta_k^a(\Phi)\ , \qquad \delta\mathcal{L} = \delta\omega_a \partial_\mu K^{a,\mu}, \qquad \delta\omega_a = \text{const.}, \tag{9.4}$$

  Appropriate form of Noether currents (for signs):

$$j^{a,\mu} = \left\{ \Delta_k^a(\Phi) \frac{\partial}{\partial(\partial_\mu \Phi_k)} \right\} \mathcal{L} - K^{a,\mu}(\Phi), \qquad \partial j^a = 0. \tag{9.5}$$

- Contractions for perturbative expansion of S-operator:

$$:\cdots \overbracket{\Phi_i \cdots \Phi_j} \cdots : \;\equiv\; (-1)^{P_{ij}} \langle 0|\, T[\Phi_i \Phi_j]\, |0\rangle \cdot \; : \ldots \Phi_{i-1}\Phi_{i+1} \cdots \Phi_{j-1}\Phi_{j+1} \cdots :, \tag{9.6}$$

  where P_{ij} = number of necessary commutations for reordering fermion operators.

  $\Rightarrow$ With (9.6) Wick theorem also holds as usual.

  Examples:

  $\diamond$ $T[\psi_1\overline{\psi}_2] \;=\; :\psi_1\overline{\psi}_2: \;+\; \overbracket{\psi_1\overline{\psi}_2} \;=\; :\psi_1\overline{\psi}_2: \;+\; \langle 0|\, T[\psi_1\overline{\psi}_2]\, |0\rangle,$

  $\diamond$
$$\begin{aligned} T[\psi_1\overline{\psi}_2\psi_3\overline{\psi}_4] \;=\; & :\psi_1\overline{\psi}_2\psi_3\overline{\psi}_4: \\ & + \; :\psi_1\overline{\psi}_2: \; \langle 0|\, T[\psi_3\overline{\psi}_4]\, |0\rangle \; + :\psi_3\overline{\psi}_4: \; \langle 0|\, T[\psi_1\overline{\psi}_2]\, |0\rangle \\ & - \; :\psi_3\overline{\psi}_2: \; \langle 0|\, T[\psi_1\overline{\psi}_4]\, |0\rangle \; - :\psi_1\overline{\psi}_4: \; \langle 0|\, T[\psi_3\overline{\psi}_2]\, |0\rangle \\ & + \; \langle 0|\, T[\psi_1\overline{\psi}_2]\, |0\rangle \, \langle 0|\, T[\psi_3\overline{\psi}_4]\, |0\rangle \\ & - \; \langle 0|\, T[\psi_1\overline{\psi}_4]\, |0\rangle \, \langle 0|\, T[\psi_3\overline{\psi}_2]\, |0\rangle\, . \end{aligned}$$

9.2 Yukawa theory

Definition of the model:

Lagrangian:

$$\begin{aligned}
\mathcal{L}(\phi,\psi,\overline{\psi}) &= \mathcal{L}_{\psi,0}+\mathcal{L}_{\phi,0}+\mathcal{L}_{\text{int}}, \\
\mathcal{L}_{\psi,0}(\psi,\overline{\psi}) &= \,:\overline{\psi}\,(\mathrm{i}\partial\!\!\!/ - m_f)\,\psi:, \quad \text{Dirac fermion of mass } m_f, \\
\mathcal{L}_{\phi,0}(\phi) &= \,:\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)-\frac{1}{2}m_\phi^2\phi^2:, \quad \text{neutral scalar field of mass } m_\phi, \\
\mathcal{L}_{\text{int}}(\phi,\psi,\overline{\psi}) &= -y:\phi\,\overline{\psi}\,\psi:, \quad \text{Yukawa interaction.}
\end{aligned} \tag{9.7}$$

Hamiltonian:

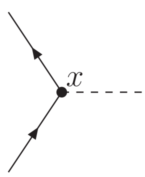
$$\mathcal{H}_{\text{int}}(\phi,\psi,\overline{\psi}) = -\mathcal{L}_{\text{int}}(\phi,\psi,\overline{\psi}), \quad \text{since no derivative involved.} \tag{9.8}$$

9.2.1 Feynman rules for the S-operator

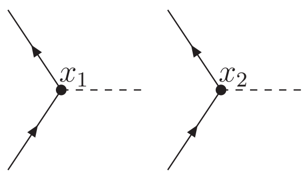
Expansion of the S-operator:

$$\begin{aligned}
S &= T\exp\left[\int \mathrm{d}^4x\ \mathrm{i}\mathcal{L}_{\text{int}}(\phi(x),\psi(x),\overline{\psi}(x))\right] \\
&= T\exp\left[-\int \mathrm{d}^4x\,\mathrm{i}y:\phi(x)\,\overline{\psi}(x)\,\psi(x):\right] \\
&= \mathbf{1}+(-\mathrm{i}y)\,T\left[\int \mathrm{d}^4x:\phi(x)\,\overline{\psi}(x)\,\psi(x):\right] \\
&\quad +\frac{1}{2}(-\mathrm{i}y)^2\,T\left[\int \mathrm{d}^4x_1\int \mathrm{d}^4x_2\ :\phi(x_1)\,\overline{\psi}(x_1)\,\psi(x_1)::\phi(x_2)\,\overline{\psi}(x_2)\,\psi(x_2):\right]+\dots
\end{aligned}$$

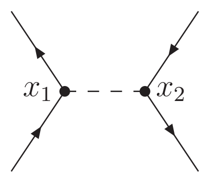
$$= \mathbf{1}+(-\mathrm{i}y)\left[\int \mathrm{d}^4x\ :\ \phi(x)\,\overline{\psi}(x)\,\psi(x):\right]$$



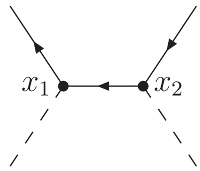
$$+\frac{1}{2}(-\mathrm{i}y)^2\left[\int \mathrm{d}^4x_1\int \mathrm{d}^4x_2\ :\phi(x_1)\,\overline{\psi}(x_1)\,\psi(x_1)\phi(x_2)\,\overline{\psi}(x_2)\,\psi(x_2):\right.$$

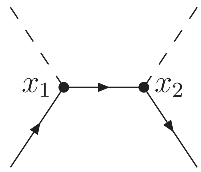


$$+\int \mathrm{d}^4x_1\int \mathrm{d}^4x_2\ :\overline{\psi}(x_1)\,\psi(x_1)\overbrace{\phi(x_1)\phi(x_2)}\overline{\psi}(x_2)\,\psi(x_2):$$



$$+\int \mathrm{d}^4x_1 \int \mathrm{d}^4x_2 \ :\phi(x_1)\,\overline{\psi}(x_1)\psi(x_1)\overline{\psi}(x_2)\,\psi(x_2)\,\phi(x_2):$$

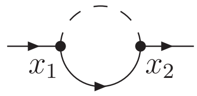




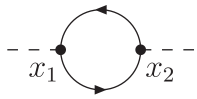
$$+\int \mathrm{d}^4x_1 \int \mathrm{d}^4x_2 \ \underbrace{:\ \phi(x_2)\,\bar{\psi}(x_2)\,\psi(x_2)\overline{\psi}(x_1)\,\psi(x_1)\,\phi(x_1):}_{\text{convenient form for contractions with external states}}$$

$$+\int \mathrm{d}^4x_1 \int \mathrm{d}^4x_2 \ :\ \overline{\psi}(x_1)\,\psi(x_1)\overline{\psi}(x_2)\,\psi(x_2)\phi(x_1)\phi(x_2):$$

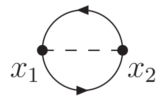
$$+\int \mathrm{d}^4x_1 \int \mathrm{d}^4x_2 \ :\ \bar{\psi}(x_2)\,\psi(x_2)\overline{\psi}(x_1)\,\psi(x_1)\phi(x_1)\phi(x_2):$$



$$-\int \mathrm{d}^4x_1 \int \mathrm{d}^4x_2 \ :\ \phi(x_1)\,\phi(x_2): \underbrace{\sum_{\alpha,\beta}\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)\psi_\beta(x_2)\bar{\psi}_\alpha(x_1)}_{=\operatorname{tr}\left[\psi(x_1)\overline{\psi}(x_2)\psi(x_2)\overline{\psi}(x_1)\right]}$$

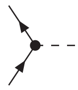


$$-\int \mathrm{d}^4x_1 \int \mathrm{d}^4x_2 \ :\phi(x_1)\phi(x_2)\psi(x_1)\overline{\psi}(x_2)\psi(x_2)\overline{\psi}(x_1):\Bigg]$$



$$+\ldots \tag{9.9}$$

Feynman rules for graphical representation of the terms $\propto y^n$:

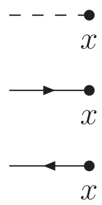
1. Draw all possible diagrams with n vertices 

   (any number of exernal lines, including disconnected diagrams).

2. Translate graphs into analytical expressions as follows:

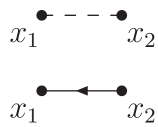
   - External lines $\hat{=}$ non-contracted fields:

$$\phi(x) = \qquad \psi(x) = \qquad \bar{\psi}(x) = \tag{9.10}$$



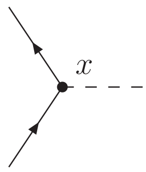
   - Internal lines $\hat{=}$ contracted fields (=propagators):

$$\phi(x_1)\phi(x_2) = \qquad \psi(x_1)\bar{\psi}(x_2) = \tag{9.11}$$



   - Vertices $\hat{=}$ interaction terms:

$$-\,\mathrm{i}y \;= \tag{9.12}$$



3. Order terms opposite to the fermion flow indicated by the arrows.

4. For each closed fermion loop take Dirac trace and multiply by (-1).

5. Integrate the sum of all terms according to

$$\frac{1}{n!}\int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n : \;\ldots\; : . \tag{9.13}$$

9.2.2 Feynman rules for S-matrix elements

Consider $n \to m$ particle process:

$$|i\rangle = a^\dagger_{A_1} \ldots a^\dagger_{A_n}\,|0\rangle \;\;, \quad \langle f| = \langle 0|\, a_{B_m} \ldots a_{B_1} \tag{9.14}$$

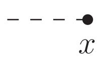
where $A_1, \ldots, B_m$ = scalar or fermion fields ϕ, ψ, $\bar{\psi}$.

$\hookrightarrow$ Only contributions from terms $\propto a^\dagger_{B_1} \ldots a^\dagger_{B_m} a_{A_n} \ldots a_{A_1}$ in S!

1. Select terms graphically: A_i, $B_j \;\hat{=}$ external lines in diagrams


x incoming fermion/outgoing antifermion: $\psi(x)$ contains a, $b^\dagger$ (9.15)


x outgoing fermion/incoming antifermion (9.16)


x incoming/outgoing real scalar (9.17)

2. Perform contractions of fields in normal-ordered products with external fields:

   Typical manipulation:

$$\psi(x)a_\sigma^\dagger(\vec{p})\,|0\rangle = \int \mathrm{d}\tilde{k}\,\sum_\tau \Big[\mathrm{e}^{-\mathrm{i}kx}u_\tau(k)\,\underbrace{a_\tau(\vec{k})a_\sigma^\dagger(\vec{p})}_{=-a_\sigma^\dagger(\vec{p})a_\tau(\vec{k})+\underbrace{\{a_\tau(\vec{k}),a_\sigma^\dagger(\vec{p})\}}_{=(2\pi^3)2p^0\delta(\vec{p}-\vec{k})\delta_{\sigma\tau}}} + ...\Big]|0\rangle \tag{9.18}$$

$$= \mathrm{e}^{-\mathrm{i}px}u_\sigma(p)\,|0\rangle + \dots \tag{9.19}$$

   Define contractions with external fermion fields:

$$\begin{aligned}
\psi(x)a_\sigma^\dagger(\vec{p})\,|0\rangle &= \mathrm{e}^{-\mathrm{i}px}u_\sigma(p)\,|0\rangle\,,\\
\bar{\psi}(x)b_\sigma^\dagger(\vec{p})\,|0\rangle &= \mathrm{e}^{\mathrm{i}px}\bar{v}_\sigma(p)\,|0\rangle\,,\\
\langle 0|\,a_\sigma(\vec{p})\bar{\psi}(x) &= \langle 0|\,\mathrm{e}^{\mathrm{i}px}\bar{u}_\sigma(p),\\
\langle 0|\,b_\sigma(\vec{p})\psi(x) &= \langle 0|\,\mathrm{e}^{-\mathrm{i}px}v_\sigma(p).
\end{aligned} \tag{9.20}$$

3. Perform the integration of $\int \mathrm{d}^4x_i$ after inserting the propagators:

$$\mathrm{i}D_F(x_1,x_2) = \phi(x_1)\phi(x_2) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}k(x_1-x_2)}\frac{\mathrm{i}}{k^2-m_\phi^2+\mathrm{i}\epsilon} = \tag{9.21}$$

x_1 x_2

$$\mathrm{i}S_F(x_1,x_2) = \psi(x_1)\bar{\psi}(x_2) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}k(x_1-x_2)}\frac{\mathrm{i}}{\not{k}-m_f+\mathrm{i}\epsilon} = \tag{9.22}$$

x_1 x_2

   $\hookrightarrow$ Momentum conservation at each vertex.

4. Calculate symmetry factor for each graph (are all 1 in this model).

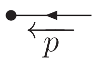
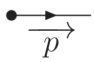
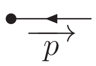
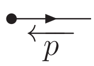
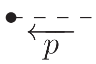
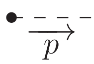
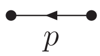
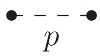
5. Determine the sign for each graph from the permutation of fermionic operators.

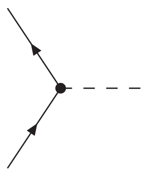
Feynman rules for the transition matrix element $\mathcal{M}_{fi}$:

1. Determine all relevant Feynman diagrams:
   - $n \to m$ scattering process $\Rightarrow$ $n + m$ external lines.
   - Order of perturbation theory $\Rightarrow$ number of loops.

2. Impose momentum conservation at each vertex.

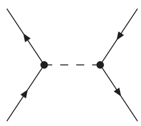
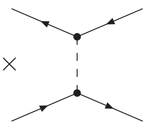
3. Insert the explicit expressions (fermionic terms ordered opposite to arrows):

	f	$u_\sigma(p)$		f	$\bar{u}_\sigma(p)$
	$\bar{f}$	$v_\sigma(p)$		$\bar{f}$	$\bar{v}_\sigma(p)$
	ϕ	1		ϕ	1
		$\dfrac{\mathrm{i}}{\not{p} - m_f + \mathrm{i}\epsilon}$			$\dfrac{\mathrm{i}}{p^2 - m_\phi^2 + \mathrm{i}\epsilon}$

 $-\mathrm{i}y$ (9.23)

4. Integrate over all loop momenta p_l via $\displaystyle\int \frac{\mathrm{d}^4 p_l}{(2\pi)^4}$.

5. For each closed fermion loop take Dirac trace and multiply by (-1). Insert a relative sign between diagrams that result from interchanging external fermion lines.

   Example:  $+ \quad (-1) \times$ 

6. The coherent sum of all diagrams yields $\mathrm{i}\mathcal{M}_{fi}$.

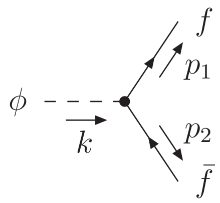
Example: the decay $\phi \to f\bar{f}$ in lowest order

$\hookrightarrow$ Practically identical to the decays $H \to f\bar{f}$ ($f = b, \tau$, etc.) of the Standard Model Higgs boson.

Process:

$$\phi(k) \;\to\; f_\sigma(p_1) \;+\; \bar{f}_\tau(p_2), \qquad \sigma, \tau = \text{helicities.} \tag{9.24}$$

Lowest-order diagram:



Amplitude:

$$\mathrm{i}\mathcal{M} \;=\; -\mathrm{i}y\,\bar{u}_\sigma(p_1)\,v_\tau(p_2). \tag{9.25}$$

$$\begin{aligned}
\Rightarrow \quad \sum_{\text{pol}} |\mathcal{M}|^2 \;&=\; y^2 \sum_{\sigma,\tau} \left(\bar{u}_\sigma(p_1)\,v_\tau(p_2)\right) \underbrace{\left(\bar{u}_\sigma(p_1)\,v_\tau(p_2)\right)^*}_{\substack{=\;(\bar{u}_\sigma(p_1)\,v_\tau(p_2))^\dagger \\ =\;\bar{v}_\tau(p_2)\,u_\sigma(p_1)}} \\
&=\; y^2 \sum_{\alpha,\beta} \underbrace{\Big(\sum_\sigma u_\sigma(p_1)_\alpha\,\bar{u}_\sigma(p_1)_\beta\Big)}_{=\,(\not{p}_1+m_f)_{\alpha\beta}} \underbrace{\Big(\sum_\tau v_\tau(p_2)_\beta\,\bar{v}_\tau(p_2)_\alpha\Big)}_{=\,(\not{p}_2-m_f)_{\beta\alpha}} \\
&=\; y^2\,\mathrm{Tr}\left\{(\not{p}_1 + m_f)(\not{p}_2 - m_f)\right\} \\
&=\; 4y^2\,(p_1p_2 - m_f^2), \qquad m_\phi^2 = k^2 = (p_1+p_2)^2 = 2m_f^2 + 2p_1p_2 \\
&=\; 2y^2\,(m_\phi^2 - 4m_f^2).
\end{aligned} \tag{9.26}$$

Partial decay width:

$$\begin{aligned}
\Gamma_{\phi\to f\bar{f}} \;&=\; \frac{1}{2m_\phi} \int \mathrm{d}\Phi_2 \sum_{\text{pol}} |\mathcal{M}|^2, \qquad \text{for phase space } \Phi_2\text{, see Exercise 7.1} \\
&=\; \frac{1}{2m_\phi}\,\frac{1}{(2\pi)^2}\,\frac{\sqrt{m_\phi^4 - 4m_\phi^2 m_f^2}}{8m_\phi^2} \underbrace{\int \mathrm{d}\Omega_1}_{=\,4\pi}\, 2y^2\,(m_\phi^2 - 4m_f^2) \\
&=\; \frac{y^2\,m_\phi}{8\pi} \left(1 - \frac{4m_f^2}{m_\phi^2}\right)^{\frac{3}{2}}.
\end{aligned} \tag{9.27}$$