# Top-pair Production at Threshold

Jan Piclum

based on (ongoing) work in collaboration with
M. Beneke, D. Boito, A. Maier, P. Marquard, A. Penin, D. Seidel, M. Steinhauser

# Outline

- Introduction
- Theoretical Framework
  - Threshold Expansion
  - Coulomb Resummation
  - Non-Relativistic Effective Theories
- Matching Coefficient of the Vector Current
- Finite-Width Effects

direct reconstruction:

$$\text{Tevatron:} \quad m_t = 173.2 \pm 0.6 \pm 0.8 \text{ GeV}$$

$$\text{CMS:} \quad m_t = 173.4 \pm 0.4 \pm 0.9 \text{ GeV}$$

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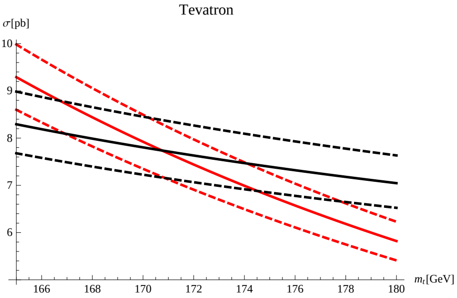
extract mass from cross section:




experimental cross section from D0

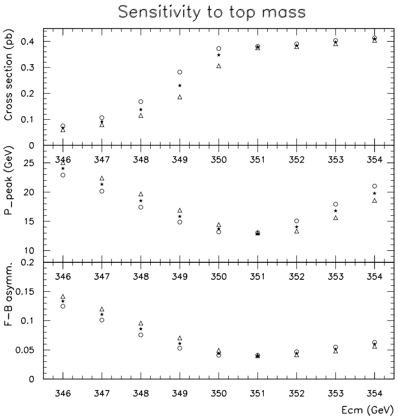
compared with

theoretical cross section computed with TOPIXS 1.0

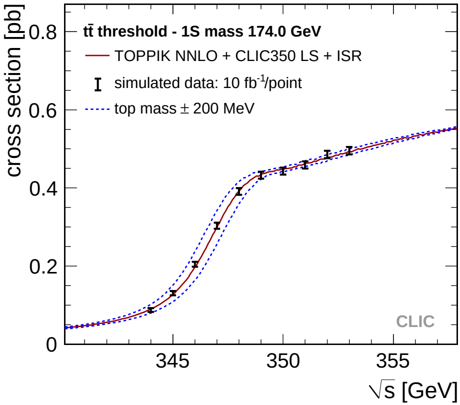
$$\rightsquigarrow m_t = 171.4^{+5.4}_{-5.7} \text{ GeV}$$

# Top Mass at ILC and CLIC

perform threshold scan:

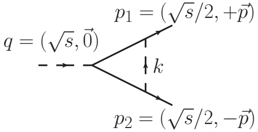



[Martinez, Miquel 2002]



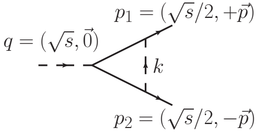

[Seidel, Simon, Tesar, Poss 2013]

$\rightsquigarrow \delta m_t \approx 100$ MeV for $\delta\sigma/\sigma \lesssim 3\%$




$$p = (0, \vec{p}), \quad p_i^2 = m^2$$

$$\sqrt{s} = E + 2m \approx 2m \quad \Rightarrow \quad v = \sqrt{\frac{E}{m}} \ll 1$$

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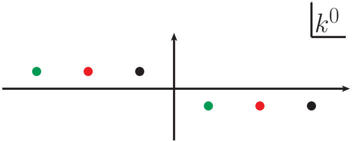
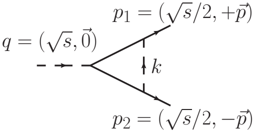
$$\int \mathrm{d}^D k \frac{q^2}{\underbrace{[(k + \frac{q}{2} + p)^2 - m^2 + i\varepsilon]}_{(k^0)^2 + q^0 k^0 - \vec{k}^2 - 2\vec{k}\cdot\vec{p} + i\varepsilon = 0}[(k - \frac{q}{2} + p)^2 - m^2 + i\varepsilon](k^2 + i\varepsilon)}$$



# Threshold Production I

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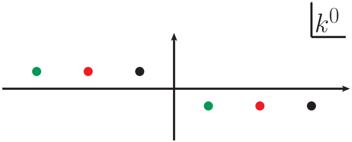


$$\to \int \frac{\mathrm{d}^{D-1}k \quad q^0}{(\vec{k}^2 + 2\vec{k}\cdot\vec{p})\,\vec{k}^2} \propto \frac{q^0}{|\vec{p}|} \propto \frac{1}{v} \gg 1$$

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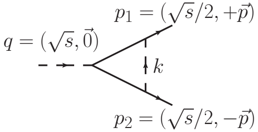


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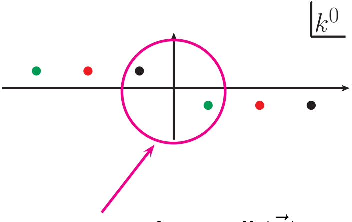


$$\rightarrow \int \frac{\mathrm{d}^{D-1}k \quad q^0}{(\vec{k}^2 + 2\vec{k}\cdot\vec{p})\,\vec{k}^2} \propto \frac{q^0}{|\vec{p}|} \propto \frac{1}{v} \gg 1$$

poles at $k^0 = \pm\frac{\vec{k}^2 + 2\vec{k}\cdot\vec{p}}{q} \mp i\varepsilon$ pinch integration contour for small $|\vec{k}|$

# Threshold Production II

relative velocity of quark-antiquark pair is small:

$$\sqrt{s} = E + 2m \approx 2m \quad \Rightarrow \quad v = \sqrt{\frac{E}{m}} \ll 1$$

- perturbation theory breaks down due to terms proportional to $\frac{\alpha_s}{v}$
  $\rightsquigarrow$ Coulomb resummation
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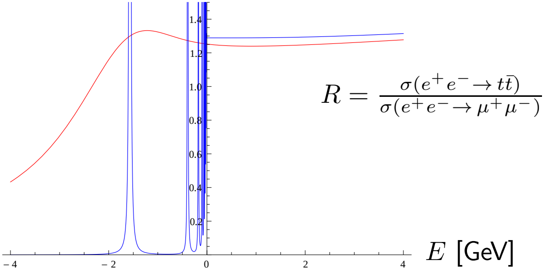
- formation of bound states below threshold
- $b\bar{b}$: bound-state resonances
- $t\bar{t}$: large width prevents existence of bound states



# Threshold Expansion

[Beneke, Smirnov 1998]

- use dimensional regularisation
- divide integration into regions
- in each region expand in small quantities
- in each region integrate over all loop momenta

see also: Smirnov *Applied Asymptotic Expansions in Momenta and Masses*, and Jantzen 2011

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scale hierarchy: $m \gg mv \gg mv^2$

relevant regions:

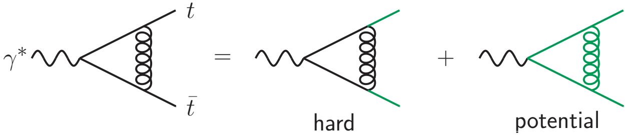
- hard: $k^0 \sim m$, $k^i \sim m$
- soft: $k^0 \sim mv$, $k^i \sim mv$
- ultrasoft: $k^0 \sim mv^2$, $k^i \sim mv^2$
- potential: $k^0 \sim mv^2$, $k^i \sim mv$

external momenta: $q = (2m + E, \vec{0}) \sim m$, $p = (0, \vec{p}) \sim mv$
quark propagator:

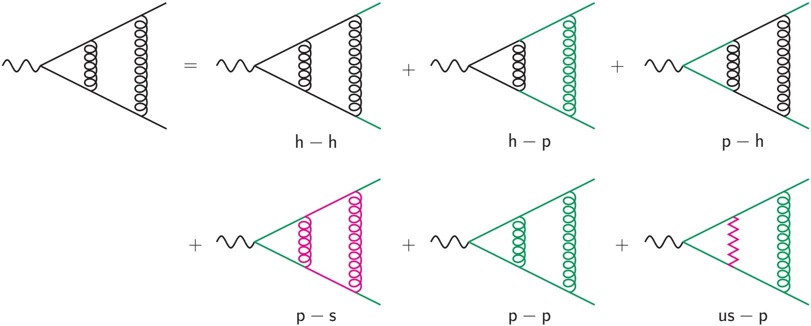
$$\frac{1}{(k + \frac{q}{2} + p)^2 - m^2} = \frac{1}{(k^0)^2 - \vec{k}^2 + q^0 k^0 - 2\vec{k}\cdot\vec{p}} = \begin{cases} \frac{1}{k^2 + k\cdot q} + \dots \\ \frac{1}{q^0 k^0} + \dots \\ \frac{1}{q^0 k^0 - \vec{k}^2 - 2\vec{k}\cdot\vec{p}} + \dots \end{cases}$$

gluon propagator in potential region:

$$\frac{1}{k^2} = \frac{1}{(k^0)^2 - \vec{k}^2} = \frac{1}{\vec{k}^2} + \dots$$



# 2-loop Example




structure of the expansion suggests effective theory treatment:

- hard loops: local operator insertions
- soft loops: instantaneous, non-local operators

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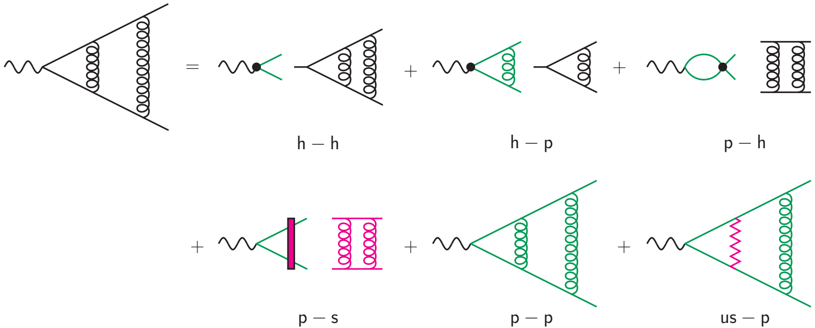


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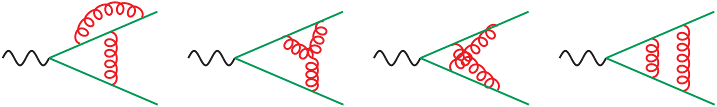
# Coulomb Resummation I

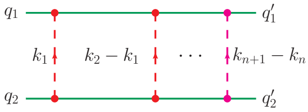


only potential region contributes to leading $(\alpha_s/v)^n$ correction:

- need quark and antiquark in the same loop
  $\rightsquigarrow$ no vertex correction
- need pinched poles
  $\rightsquigarrow$ no crossed boxes

$\rightsquigarrow$ only in ladder diagrams can all loop momenta be potential
$\rightsquigarrow$ resum potential region of ladder diagrams

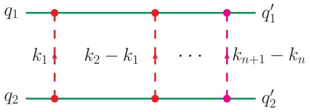
# Coulomb Resummation II




$$q_{1,2} = (m + \tfrac{E}{2}, \pm\vec{p})$$

$$q'_{1,2} = (m + \tfrac{E}{2}, \pm\vec{p}')$$

$$k_{n+1} = (0, \vec{p}' - \vec{p}),\ k_{n=0} = 0$$

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$$H(\vec{p}, \vec{p}\,'; E)$$

$$= \sum_{n=0}^{\infty} \left[ \prod_{j=1}^{n} \int \frac{\mathrm{d}^D k_j}{(2\pi)^D} \left( \frac{-i4\pi\alpha_s C_F}{(\vec{k}_j - \vec{k}_{j-1})^2} \right) \frac{i}{\frac{E}{2} + k_j^0 - \frac{(\vec{p}+\vec{k}_j)^2}{2m} + i\varepsilon} \frac{-i}{\frac{E}{2} - k_j^0 - \frac{(\vec{p}+\vec{k}_j)^2}{2m} + i\varepsilon} \right]$$

$$\times \left( \frac{-i4\pi\alpha_s C_F}{(\vec{p}\,' - \vec{p} - \vec{k}_n)^2} \right)$$

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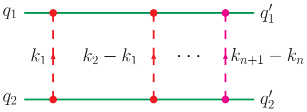


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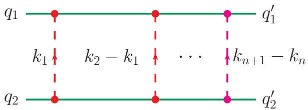
$$\times \left( \frac{-i4\pi\alpha_s C_F}{(\vec{p}\,' - \vec{p} - \vec{k}_n)^2} \right)$$

$$= i \sum_{n=0}^{\infty} \left[ \prod_{j=1}^{n} \int \frac{\mathrm{d}^{D-1} \vec{k}_j}{(2\pi)^{D-1}} \left( \frac{-4\pi\alpha_s C_F}{(\vec{k}_j - \vec{k}_{j-1})^2} \right) \frac{1}{E - \frac{(\vec{p}+\vec{k}_j)^2}{m}} \right] \left( \frac{-4\pi\alpha_s C_F}{(\vec{p}\,' - \vec{p} - \vec{k}_n)^2} \right)$$

# Coulomb Resummation III

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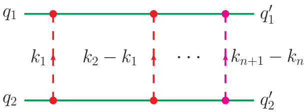
$$\tilde{G}(\vec{p}, \vec{p}\,'; E) = -\frac{(2\pi)^{D-1}\delta^{(D-1)}(\vec{p}\,' - \vec{p})}{E - \frac{\vec{p}^2}{m}} + \frac{1}{E - \frac{\vec{p}^2}{m}}\, iH(\vec{p}, \vec{p}\,'; E)\, \frac{1}{E - \frac{\vec{p}\,'^2}{m}}$$

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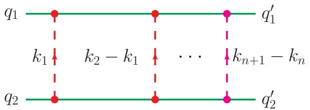


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$$\left( \frac{\vec{p}^2}{m} - E \right) \tilde{G}(\vec{p}, \vec{p}'; E) + \int \frac{\mathrm{d}^{D-1}\vec{k}}{(2\pi)^{D-1}} \left( \frac{-4\pi\alpha_s C_F}{\vec{k}^2} \right) \tilde{G}(\vec{p} + \vec{k}, \vec{p}'; E) = (2\pi)^{D-1}\delta^{(D-1)}(\vec{p}' - \vec{p})$$

$$\left( -\frac{\vec{\nabla}_r^2}{m} - \frac{\alpha_s C_F}{r} - E \right) G(\vec{r}, \vec{r}'; E) = (2\pi)^{D-1}\delta^{(D-1)}(\vec{r}' - \vec{r})$$

# Effective Theories

scale hierarchy: $m \gg mv \gg mv^2 \gg \Lambda_{\mathrm{QCD}}$

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hard subgraphs become point-like vertices

NRQCD — contains non-relativistic fields

↓ integrate out soft modes: $k^0 \sim k^i \sim mv$
soft subgraphs become instantaneous, non-local vertices

PNRQCD — contains potential quarks and ultrasoft gluons

# Non-Relativistic QCD

integrating out the hard region $\rightsquigarrow$ NRQCD

[Caswell, Lepage 1986; Lepage et al. 1992; Bodwin, Braaten, Lepage 1994; Kinoshita, Nio 1996]

$$\begin{aligned}\mathcal{L}_{\mathrm{NRQCD}} &= \psi^\dagger\left(iD^0 + \frac{\mathbf{D}^2}{2m} + \dots\right)\psi - \frac{d_1\, g_s}{2m}\psi^\dagger\sigma\cdot\mathbf{B}\psi \\ &+\psi^\dagger\left[\frac{d_2\, g_s}{8m^2}\left(\mathbf{D}\cdot\mathbf{E}-\mathbf{E}\cdot\mathbf{D}\right) + \frac{d_3\, g_s}{8m^2}\sigma\cdot\left(\mathbf{D}\times\mathbf{E}-\mathbf{E}\times\mathbf{D}\right)\right]\psi \\ &+\dots + \mathcal{L}_{\mathrm{antiquark}} + \mathcal{L}_{4-\mathrm{quark}} + \mathcal{L}_{\mathrm{light}}\end{aligned}$$

coupling to external photon: $\bar{Q}\gamma^i Q = c_v\,\psi^\dagger\sigma^i\chi + d_v\,\psi^\dagger\frac{\mathbf{D}^2}{6m^2}\sigma^i\chi + \dots$

- expansion in heavy-quark mass
- coefficients have to be determined by matching to QCD

# Potential NRQCD

integrating out the soft region $\rightsquigarrow$ PNRQCD

[Pineda, Soto 1997; Beneke, Signer, Smirnov 1999; Brambilla et al. 1999]

$$\begin{aligned}\mathcal{L}_{\mathrm{PNRQCD}} &= \psi^\dagger\left(i\partial^0 + g_s A^0(t,\mathbf{0}) + \frac{\partial^2}{2m} + \dots\right)\psi \\ &+ \int \mathrm{d}^3\mathbf{r}\,[\psi^\dagger\psi](x+\mathbf{r})\left(-\frac{\alpha_s C_F}{r} + \delta V(\mathbf{r})\right)[\chi^\dagger\chi](x) \\ &- g_s\psi^\dagger\mathbf{x}\cdot\mathbf{E}(t,\mathbf{0})\psi + \text{antiquark terms} + \dots\end{aligned}$$

- definite power-counting in $v$
- contains non-local interactions $\rightsquigarrow$ potentials
- LO Coulomb potential is part of LO Lagrangian
  $\rightsquigarrow$ Coulomb propagator
- ultrasoft fields are multipole expanded

the potential in momentum space ($\mathbf{q} = \mathbf{p}' - \mathbf{p}$):

$$\tilde{V}(\mathbf{p}, \mathbf{p}') =$$

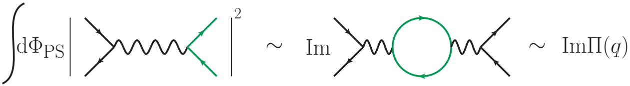
$$-\frac{4\pi\alpha_s C_F}{\mathbf{q}^2}\,\mathcal{V}_C + \frac{4\pi^3\alpha_s C_F}{m|\mathbf{q}|}\,\mathcal{V}_{1/m} + \frac{2\pi\alpha_s C_F}{m^2}\,\mathcal{V}_\delta$$

$$-\frac{2\pi\alpha_s C_F}{m^2}\frac{\mathbf{p^2} + \mathbf{p'^2}}{\mathbf{q}^2}\mathcal{V}_p - \frac{3\pi\alpha_s C_F}{2m^2\mathbf{q}^2}\left([\sigma^i, \sigma^j]q^i p^j \otimes 1 - 1 \otimes [\sigma^i, \sigma^j]q^i p^j\right)\mathcal{V}_{so}$$

$$+\frac{\pi\alpha_s C_F}{4m^2\mathbf{q}^2}[\sigma^i, \sigma^j]q^j \otimes [\sigma^i, \sigma^k]q^k\,\mathcal{V}_{hf} - \frac{\pi\alpha_s C_F}{4m^2}[\sigma^i, \sigma^j] \otimes [\sigma^i, \sigma^j]\,\mathcal{V}_s + \ldots$$

$\mathcal{V}_i$ are series in $\alpha_s$ and depend on NRQCD matching coefficients $d_i$

use optical theorem:




compute polarisation function in PNRQCD: $\Pi(q) \sim c_v^2 \, G(\vec{0},\vec{0};E)$

- compute hard matching coefficients: $d_i$, $c_v$, $d_v$
- compute potential: $\mathcal{V}_i$
- solve Schrödinger equation
- compute ultrasoft corrections

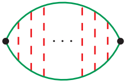
power counting:

$$\sum_n \left(\frac{\alpha_s}{v}\right)^n \times \Big\{1;\ \underbrace{\alpha_s, v}_{\text{NLO}};\ \underbrace{\alpha_s^2, \alpha_s v, v^2}_{\text{NNLO}};\ \ldots\Big\}$$

# Computing the Green's Function

Coulomb Green's function at the origin:

$$G_C(E) = \langle 0 | \frac{1}{H_0 - E} | 0 \rangle = \langle 0 | G^{(0)} | 0 \rangle$$



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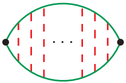
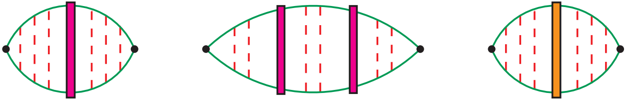


corrections due to higher order potential can be computed perturbatively:

$$\begin{aligned} \delta G(E) &= \langle 0 | \frac{1}{H_0 + \delta V - E} | 0 \rangle - G_C(E) \\ &= -\langle 0 | G^{(0)} \delta V^{(1)} G^{(0)} | 0 \rangle \\ &\quad + \langle 0 | G^{(0)} \delta V^{(1)} G^{(0)} \delta V^{(1)} G^{(0)} | 0 \rangle - \langle 0 | G^{(0)} \delta V^{(2)} G^{(0)} | 0 \rangle + \dots \end{aligned}$$



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  - potential contributions [Beneke, Kiyo, Schuller; Kniehl, Penin, Smirnov, Steinhauser]
  - 3-loop static potential [Anzai, Kiyo, Sumino; Smirnov, Smirnov, Steinhauser]
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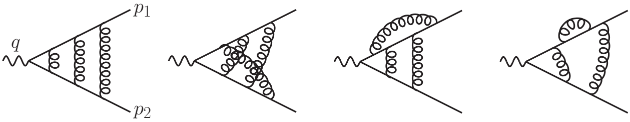
- electroweak corrections [Guth, Kühn; Eiras, Steinhauser; Kiyo, Seidel, Steinhauser]
- finite-width effects [Fadin, Khoze; Hoang, Reißer, Ruiz-Femenía; Beneke, Jantzen, Ruiz-Femenía; Penin, JP]

# The Vector Current

coupling to external photon:

$$\bar{Q}\gamma^i Q = c_v\, \psi^\dagger \sigma^i \chi + d_v\, \psi^\dagger \frac{\mathbf{D}^2}{6m^2} \sigma^i \chi + \ldots$$

- determine $c_v$ and $d_v$ by onshell matching of NRQCD to QCD
- NNNLO requires 3-loop result for $c_v$ and 1-loop result for $d_v$
- only hard region of QCD diagrams contributes

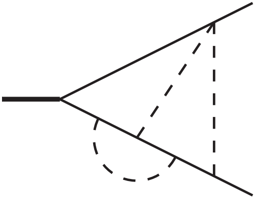



$$\begin{aligned} q^2 &= 4m^2 \\ p_i^2 &= m^2 \end{aligned}$$

# Automated Setup

- diagrams generated with `QGRAF` [Nogueira]
- mapped onto 78 topologies with `q2e` and `exp` [Harlander, Seidensticker, Steinhauser]
- $\gamma$ algebra is done in `FORM` [Vermaseren]
- reduction to master integrals with `Crusher` [Marquard, Seidel]
- identify master integrals from different topologies
  $\rightsquigarrow \mathcal{O}(100)$ master integrals

- simple ones known analytically
- some computed with Mellin-Barnes method
- most complicated ones are computed with `FIESTA` [Smirnov, Tentyukov]
- use basis transformation to check final result



$$= \frac{e^{3\epsilon\gamma_E}}{m_t^4}\left(\frac{\mu^2}{m_t^2}\right)^{3\epsilon}\left(\frac{0.411236(3)}{\epsilon^2}+\frac{3.4860(1)}{\epsilon} + 34.520(2) + 339.68(4)\,\epsilon + \mathcal{O}(\epsilon^2)\right)$$

# Result

residue of polarisation function:

$$\Pi(q^2) \overset{E \to E_n}{\sim} \frac{Z_n}{E_n - E - i\varepsilon}$$

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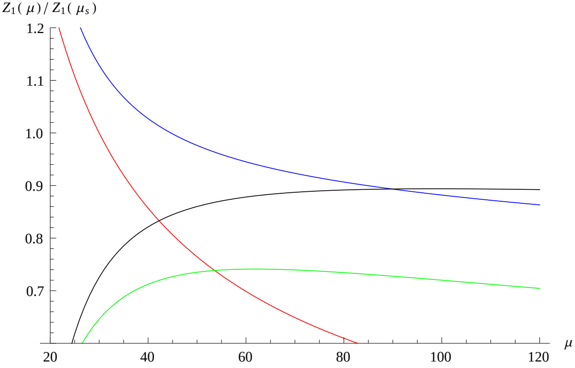
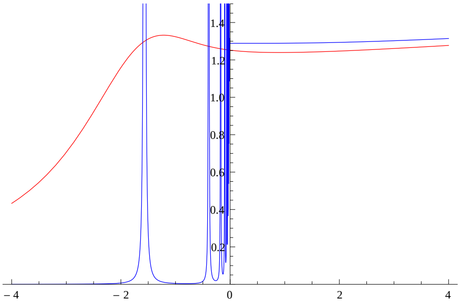


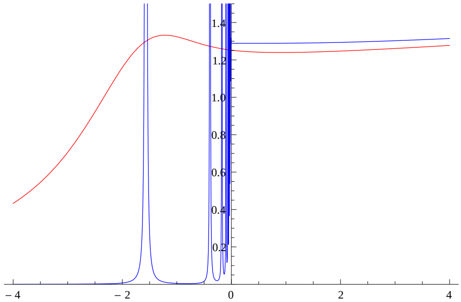

- LO
- NLO
- NNLO
- NNNLO

# Finite-Width Effects I




$$G_C^{\overline{\mathrm{MS}}}(E) = \frac{m_t^2 \alpha_s C_F}{4\pi} \left( \ln \frac{\lambda\mu}{m_t \alpha_s C_F} + \frac{1}{2} - \gamma_E - \frac{1}{2\lambda} - \Psi(1-\lambda) \right) , \quad \lambda = \frac{m_t \alpha_s C_F}{2\sqrt{-m_t E}}$$

- bound-state resonances are due to poles of $\Psi$ for $\lambda \to n$




$$G_C^{\overline{\mathrm{MS}}}(E) = \frac{m_t^2 \alpha_s C_F}{4\pi} \left( \ln \frac{\lambda \mu}{m_t \alpha_s C_F} + \frac{1}{2} - \gamma_E - \frac{1}{2\lambda} - \Psi(1-\lambda) \right), \quad \lambda = \frac{m_t \alpha_s C_F}{2\sqrt{-m_t E}}$$

- bound-state resonances are due to poles of $\Psi$ for $\lambda \to n$
- introduce width at LO: $E \to E + i\Gamma_t$ [Fadin, Khoze 1987]
- resonances are smoothed out into broad peak

# Finite-Width Effects II

Coulomb Green's function is divergent:

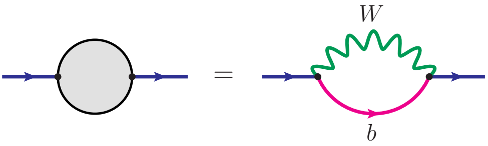
$$G_C(E) = \frac{m_t^2 \alpha_s C_F}{4\pi} \left( \frac{1}{4\epsilon} + \ln \frac{\lambda\mu}{m_t \alpha_s C_F} + \frac{1}{2} - \gamma_E - \frac{1}{2\lambda} - \Psi(1-\lambda) + \mathcal{O}(\epsilon) \right)$$

- divergence only contributes to real part at LO
- NNLO corrections introduce terms $\sim E\, G_C(E)$
  $\rightsquigarrow$ divergences in imaginary part after energy shift $\sim \Gamma_t\, \mathrm{Im} G_C(E)$
- divergences have to be cancelled by finite-width contributions

[Hoang, Reißer; Beneke, Kiyo]

finite width is due to imaginary part of top selfenergy:

$\Sigma(p, m_t) =$




$+ \dots$

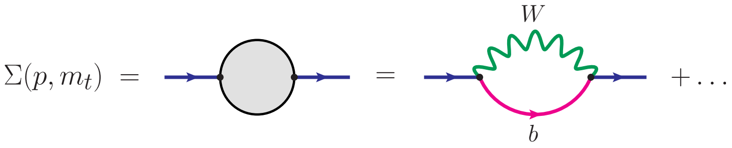
unstable particle: $\mathrm{Im}\Sigma = \frac{1}{2}\Gamma_t(p^2 = m_t^2) \; + \; \mathcal{O}(p^2 - m_t^2)$

resum selfenergy into propagator:

$$\frac{i}{\frac{E}{2} + k^0 - \frac{(\vec{p}+\vec{k})^2}{2m}} \to \frac{i}{\frac{E}{2} + i\frac{\Gamma_t}{2} + k^0 - \frac{(\vec{p}+\vec{k})^2}{2m}}$$

# Origin of Finite-Width Corrections

finite width is due to imaginary part of top selfenergy:

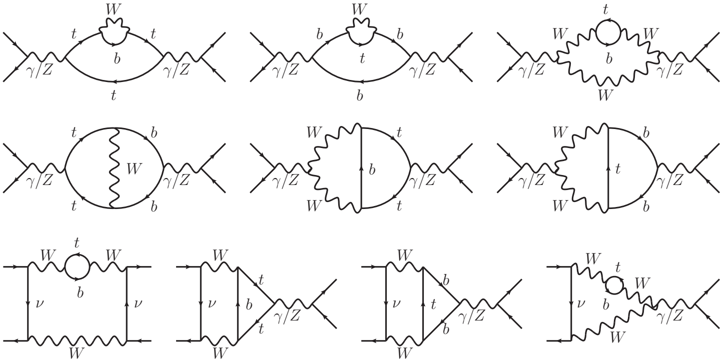



unstable particle: $\mathrm{Im}\Sigma = \frac{1}{2}\Gamma_t(p^2 = m_t^2) + \mathcal{O}(p^2 - m_t^2)$ ?

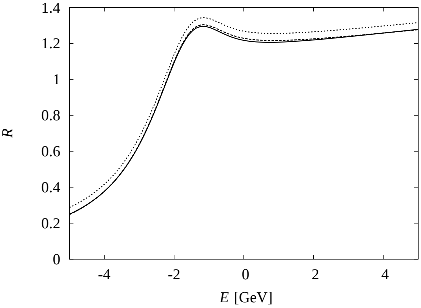
resum selfenergy into propagator:

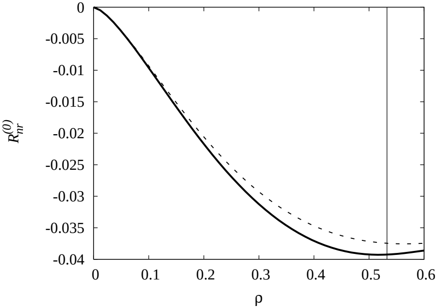
$$\frac{i}{\frac{E}{2} + k^0 - \frac{(\vec{p}+\vec{k})^2}{2m}} \to \frac{i}{\frac{E}{2} + i\frac{\Gamma_t}{2} + k^0 - \frac{(\vec{p}+\vec{k})^2}{2m}}$$

# Finite-Width at NLO

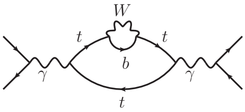



- consider only $Wtb$ cuts: $e^+e^- \to tt^* \to tWb$
- compute diagrams as expansion in $\rho = 1 - m_W/m_t$
- expansion can be formulated in terms of effective theory




- NLO contribution shifts LO result by $-3\%$
- correction mainly due to single diagram



# Summary

- threshold production allows for very precise and theoretically clean determination of top-quark mass
- NNNLO calculation is almost done
- size of correction still has to be understood:
  - resummation of logarithms
  - renormalons
- additional application: bottom-quark sum rules